INTRINSIC LINKING OF COMPLETE PARTITE GRAPHS

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Abstract. We classify complete partite graphs with respect to intrinsic linking. In Adam's *The Knot Book*, it is conjectured that on removing a vertex from an intrinsically knotted graph, one obtains an intrinsically linked graph. We verify this conjecture for the class of complete partite graphs.

1. Introduction

We say that a graph is intrinsically knotted (respectively, linked) if every tame embedding of the graph in $\mathbb{R}^3$ contains a non-trivially knotted cycle (respectively, pair of non-trivially linked cycles). In *The Knot Book* [A], Adams poses the following "unsolved question:" *Is it true that if G [a graph] is intrinsically knotted and any one vertex and the edges coming into it are removed, the remaining graph is intrinsically linked?* Although Foisy [Fo] has shown that the conjecture does not hold in general, it remains open for specific types of graphs. In this paper we demonstrate the validity of the conjecture for the class of complete partite graphs:

Theorem 1. *If G is an intrinsically knotted complete partite graph, and any one vertex and the edges coming into it are removed, the remaining graph is intrinsically linked.*

The proof rests on the classification of complete partite graphs with respect to intrinsic knotting and linking. Fleming [Fl] recently announced a knotting classification. This is complemented by our own classification in terms of intrinsic linking:

Theorem 2. *The complete k-partite graphs are classified with respect to intrinsic linking according to the following table.*

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k	1	2	3	4	5	≥ 6
linked	6	4,4	3,3,1 4,2,2	2,2,2,2 3,2,1,1	2,2,1,1,1 3,1,1,1,1	*All*
not linked	5	n,3	3,2,2 n,2,1	2,2,2,1 n,1,1,1	2,1,1,1,1	*None*

TABLE 1. Intrinsic linking of complete k-partite graphs.

Remark: For each k, the table includes minor-minimal examples of intrinsically linked complete k-partite graphs and maximal graphs which are not intrinsically linked. For example, any complete 2-partite graph which contains $K_{4,4}$ as a minor is intrinsically linked. On the other hand, any complete 2-partite graph which is a minor of a $K_{n,3}$ ($n \geq 3$. Our convention is to write the parts in decreasing order.) is not linked. Thus, $K_{l,m}$ is intrinsically linked iff $l \geq 4$ and $m \geq 4$.

2. PROOFS

We begin by proving our classification theorem, Theorem 2. This is followed by a proof of Theorem 1. In the remainder of the paper, we will frequently omit the words "complete", referring simply to partite graphs, and "intrinsically", referring simply to linked or knotted graphs.

Theorem 2. *The complete k-partite graphs are classified with respect to intrinsic linking according to Table 1.*

Proof:
<u>Linked:</u>

Let us demonstrate that the graphs labelled "linked" in Table 1 are in fact intrinsically linked.

Note that $K_{n_1+n_2,n_3,\dots,n_k}$ is a minor of $K_{n_1,n_2,\dots,n_k}$ since combining the n_1 and n_2 parts only involves removing edges between vertices in the n_1 part and the n_2 part. More generally, any graph obtained by combining parts in this way will be a minor of the original graph. This is an important construction since a graph containing a linked minor is itself linked [MRS]. Consequently, the minors of an unlinked graph are also not linked.

Conway and Gordon [CG] proved that K_6 is linked. Any k-partite graph with $k \geq 6$ contains $K_6 = K_{1,1,1,1,1,1}$ as a minor and is therefore linked. For the remaining k, we appeal to work of Robertson, Seymour, and Thomas [RST] who showed that a graph is is intrinsically linked iff it contains as a minor one of the seven graphs in the Petersen family. In particular, $K_{3,3,1}$ and $K_{4,4}$ with one edge removed are both in this family. By combining parts, we see that, for $2 \leq k \leq 5$, one of these two is a minor of each of the "linked" graphs in Table 1.

Not Linked:

Fleming's [Fl] Theorem 2.1 is another important characterisation of intrinsically linked graphs. In our context it states $K_{n_1,n_2,\ldots n_k,1}$ is intrinsically linked iff $K_{n_1,n_2,\ldots n_k}$ is non-planar. For each of the "not linked" examples in Table 1 which involve a part with a single vertex, the corresponding graph obtained by removing that vertex is planar. Therefore, by Fleming's theorem, these graphs are not intrinsically linked. Since a cycle requires at least three vertices, K_5 has no disjoint pair of cycles and is therefore not linked. The remaining "not linked" graphs in Table 1, $K_{n,3}$ and $K_{3,2,2}$, are, respectively, minors of the unlinked graphs $K_{n,2,1}$ and $K_{2,2,2,1}$. □

Theorem 1. *If G is an intrinsically knotted complete partite graph, and any one vertex and the edges coming into it are removed, the remaining graph is intrinsically linked.*

Proof: For the reader's convenience, we reproduce Fleming's [Fl] classification of knotted partite graphs as Table 2 below.

k	1	2	3	4	5	6	≥ 7
knotted	7	5,5	3,3,3 4,3,2 4,4,1	3,2,2,2 4,2,2,1 3,3,2,1 3,3,1,1	2,2,2,2,1 3,2,2,1,1 3,2,1,1,1	2,2,1,1,1,1 3,1,1,1,1,1	All
not knotted	6	4,4	3,3,2 n,2,2 n,3,1	2,2,2,2 4,2,1,1 3,2,2,1 n,2,1,1 n,1,1,1	2,2,2,1,1 2,2,1,1,1 n,1,1,1,1	2,1,1,1,1,1	None

TABLE 2. Intrinsic knotting of k-partite graphs.

It suffices to verify the theorem for minor-minimal examples of knotted k-partite graphs, $k = 1, 2, 3, \ldots$.

- $k = 1$: A complete graph K_n is intrinsically knotted iff $n \geq 7$. Removing a vertex from K_7 produces K_6 which is intrinsically linked.
- $k = 2$: $K_{5,5}$ is the minor-minimal knotted 2-partite graph. (Note that $K_{n,4}$ is not intrinsically knotted for $n \geq 5$. This is implicit in [Fl].) Removing a vertex from $K_{5,5}$ yields $K_{5,4}$ which is intrinsically linked.

$k = 3$: There are three minor-minimal knotted 3-partite graphs: $K_{3,3,3}$, $K_{4,3,2}$, and $K_{4,4,1}$. Removing a vertex from one of these results in one of the following linked graphs: $K_{4,4}$, $K_{3,3,2}$, $K_{4,2,2}$, or $K_{4,3,1}$.

$k = 4$: Here the minimal knotted graphs are $K_{3,2,2,2}$, $K_{3,3,1,1}$ and $K_{4,2,2,1}$ (The graph $K_{3,3,2,1}$ listed by Fleming [Fl] is redundant as it includes $K_{3,3,1,1}$ as a minor.) Removing a vertex from any of these we obtain one of the linked graphs $K_{3,3,1}$, $K_{4,2,2}$, $K_{2,2,2,2}$, $K_{3,2,1,1}$, $K_{3,2,2,1}$, or $K_{4,2,1,1}$.

$k = 5$: In this case we must check the knotted graphs $K_{2,2,2,2,1}$ and $K_{3,2,1,1,1}$. (Fleming's [Fl] $K_{3,2,2,1,1}$ is redundant.) Taking a vertex from either of these results in a linked graph: $K_{2,2,2,2}$, $K_{3,2,1,1}$, $K_{2,2,1,1,1}$, $K_{2,2,2,1,1}$, or $K_{3,1,1,1,1}$.

$k = 6$: Here there are two knotted graphs: $K_{2,2,1,1,1,1}$ and $K_{3,1,1,1,1,1}$. After a vertex is deleted, we're left with one of these linked graphs: $K_{2,2,1,1,1}$, $K_{3,1,1,1,1}$, or $K_{2,1,1,1,1,1}$.

$k \geq 7$: All such graphs are intrinsically knotted. On removing a vertex, we obtain an l-partite graph where $l \geq 6$. All such graphs are intrinsically linked.

□

References

[A] C.C. Adams, *The Knot Book*, W.H. Freeman and Company (1994).

[CG] J.H. Conway & C.McA. Gordon, 'Knots and links in spatial graphs,' *J. Graph Theory* **7** (1983), 445–453.

[Fl] T. Fleming, 'Intrinsically linked graphs, Kuratowski minors, and embeddings in surfaces,' (preprint).

[Fo] J. Foisy, 'A newly recognized intrinsically knotted graph,' *J. Graph Theory* **43** (2003), 199–209.

[MRS] Motwani, R. Raghunathan, and Saran, H. 'Constructive results from graph minors: Linkless embeddings,' *29th Annual Symposium on Foundations of Computer Science, IEEE* (1998), 398–409.

[RST] N. Robertson, P. Seymour, R. Thomas, 'Sachs' linkless embedding conjecture,' *J. Combin. Theory Ser. B* **64** (1995), 185–227.

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