THE STABILITY OF NON-WALRASIAN PROCESSES:
TWO EXAMPLES

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As a non-Walrasian system tracks through the phase space, the differential equations which govern its motion will typically change as the system crosses certain borders. This increases the complexity of the stability problem considerably. In the present paper we find that some straightforward modifications to Lyapunov's method render the problem tractable. These methods are derived, and their use is illustrated in the case of two different systems which have trading out of equilibrium.

1. THOUGH ECONOMISTS HAVE BEEN INTERESTED in the stability on non-Walrasian systems at least since Clower's paper¹ over a decade ago, we have yet to get very far with the inquiry. There may be a number of reasons for this, but the most important seems to be that we have not yet fully appreciated the differences between the methods of analysis suitable for studying the stability of Walrasian and non-Walrasian systems. It is widely known that the primary distinction between the two systems is that quantities actually traded enter as arguments in the non-Walrasian excess demand functions. These quantities will sometimes be demand quantities and sometimes supply quantities, depending upon the overall state of the markets—but then this implies that the excess demand functions themselves will be changing as the system moves through time and that the system itself is not everywhere differentiable. Take, for example, an output supply function which depends upon the actual quantity of labor hired. If the actual quantity hired is the lesser of the quantities supplied and demanded, then under the usual assumptions the partial derivative of output supply with respect to the price of labor will sometimes be positive, sometimes negative, and sometimes non-existent, depending upon whether the demand for labor is greater than, less than, or equal to the supply. What we have in effect is a dynamic system which has its endogenous variables sometimes governed by one set of equations and sometimes by another, with the overall system lacking differentiability at the points of changeover. This much is fairly clear, but the methods which can be used to study such systems have, with few exceptions,² yet to be seriously explored.

In the present paper we are interested in finding modifications to Lyapunov theorems which will render them suitable for the study of non-Walrasian systems. Two such modifications are found and their usefulness in studying non-Walrasian systems is illustrated by means of some relatively simple economic examples.

¹ See Clower [5].

² Hal Varian [10] has recently shown that the artful construction of "virtual" systems together with the Poincare-Hopf theorem can be used to study such systems. His paper is discussed at some length in Section 5. See also M. Aoki [1, pp, 202ff].
Consider the systems of autonomous differential equations

\[
\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n) \quad (i = 1, \ldots, n),
\]

or simply

\[
\dot{x} = f(x),
\]

where \( \dot{x}_i = dx_i/dt \) and \( x = (x_1, \ldots, x_n) \). Let \( f(0) = 0 \), that is let the origin be an equilibrium to \( (a) \), and let \( G \) be a region of the phase space which contains 0. We assume that the functions \( f(x) \) are continuous and satisfy the Lipschitz conditions in \( G \). And further, we assume that the origin is an isolated equilibrium.

We state the following theorem without proof.\(^3\)

**THEOREM 1:** If (i) there exists a continuous positive definite function \( V(x) \), with (ii) \( V(x) < 0 \) outside \( M \), and (iii) \( V(x) \leq 0 \) on \( M \), where (iv) \( M \) is a set containing no entire trajectories other than the null trajectory, then the origin is asymptotically stable.

Figure 1 is offered to help fix ideas. The ovals surrounding the origin are level surfaces of the function \( V(x) \). Since \( V(x) \) is positive definite, and since the origin is an isolated equilibrium, the level surfaces of \( V(x) \) sufficiently “close” to 0 are

\[ C_1 > C_0 \]

\( \text{FIGURE 1} \)

\(^3\) The theorem is an extension of Lyapunov's well-known theorem on asymptotic stability. See Barbashin [4, p. 28] for the proof. See also La Salle and Lefschetz [8, pp. 58–59].
closed with respect to the origin; and further, the surface \( V(x) = c_0 \) lies within the surface \( V(x) = c_1 \), where \( c_1 > c_0 \). For simplicity, we let \( M \) be the dashed line running through the origin. Except for the point \( \alpha \), \( V \) is decreasing everywhere along the half-trajectory emanating from \( q \)—that is, the trajectory moves from the outside to the inside of the level surfaces everywhere but at \( \alpha \). The trajectory at \( \alpha \) is momentarily tangent to the level surface, thus at \( \alpha \) we have \( \dot{V}(x) = 0 \). But since \( \alpha \neq 0 \) cannot be an entire trajectory, the path must run out of \( \alpha \) and continue through ever lower level surfaces to 0. Essentially, this theorem is just like the more conventional Lyapunov theorem on asymptotic stability, with the exception that it also shows that \( \dot{V}(x) = 0 \) on \( M \) is not destabilizing.

We suppose that (a) is asymptotically stable by Theorem 1. That is, we make the following assumption.

\[\text{ASSUMPTION (a): (i) There is a continuous, positive definite function } V(x). \text{ (ii) } \dot{V}(x) = \sum_{i=1}^{n} (\partial V/\partial x_i) f_i(x) < 0, \text{ for } x \not\in M^a, \text{ where } f_i(x) = x_i \text{ and } M^a \text{ is system (a)'s set } M \text{ as defined above. (iii) } \dot{V}(x) = \sum_{i=1}^{n} (\partial V/\partial x_i) f_i(x) \leq 0, x \in M^a. \text{ (iv) } M^a \text{ contains no entire trajectories other than the origin.}\]

Now suppose we have another system:

\( \dot{x} = g(x), \)

which is defined on the same region \( G \) of the phase space, has the same isolated critical point, and can be shown to be asymptotically stable using Theorem 1 and the same function \( V(x) \). That is, we make Assumption (b), which is the same as Assumption (a) except that every \( f_i(x) \) is replaced by \( g_i(x) \) and \( M^a \) by \( M^b \).

We are interested now in the stability of a third dynamical system which is a sort of hybrid formed from the other two. More specifically, we suppose that the dynamic equations which govern \( \dot{x}(t) \) can change as \( x(t) \) moves from one part of \( G \) to another. Thus we assume

\[\dot{x} = \begin{cases} f(x), & x \in S^a, \\ g(x), & x \in S^b, \\ f(x) = g(x), & x \in S^a \cap S^b = W, \end{cases}\]

where \( S^a(S^b) \) is the subset of \( G \) within which system (a) (system (b)) provides the dynamic force for \( \dot{x} \). We assume that \( S^a \) and \( S^b \) are a covering, i.e., \( S^a \cup S^b = G \). We do not assume that \( S^a \) and \( S^b \) are disjoint, only that in the "switching area" \( W = S^a \cap S^b \), \( f(x) = g(x) \)—of course, \( S^a \cap S^b = \emptyset \) is not ruled out.

It is not difficult to show the following theorem.

**THEOREM 2:** The "switching system" (c) satisfies the assumptions of Theorem 1 and is, therefore, asymptotically stable.

**PROOF:** (i) Nothing is changed with regard to the function \( V(x) \) itself, since it is simply defined on \( x \), i.e., \( V(x) \) is still continuous and positive definite by the

\(^4\) For a discussion, see Leighton [9, p. 272] and Barbashin [4, p. 22].
previous assumptions. (ii) Outside $M = M^a \cup M^b$ we have

$$V(x) = \begin{cases} 
\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) < 0, & x \in S^a \setminus M, \\
\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} g_i(x) < 0, & x \in S^b \setminus M, \\
\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} g_i(x) < 0, & x \in W \setminus M.
\end{cases}$$

(iii) On $M$ we have

$$V(x) = \begin{cases} 
\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \leq 0, & x \in M^a, \\
\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} g_i(x) \leq 0, & x \in M^b, \\
\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} g_i(x) \leq 0, & x \in M^a \cap M^b.
\end{cases}$$

(iv) If neither $M^a$ nor $M^b$ contains an entire trajectory other than the origin, $M$ cannot contain such a trajectory.

Again, a picture may be helpful. The ovals around the origin in Figure 2 are the level surfaces of $V(x)$. The shaded area in the northeast represents the subspace $S^b$, and the area southwest of this is $S^a$. The switching line $W$ marks the boundary.

![Figure 2](image-url)

Beginning at $q$, the motion of the system is governed by differential system (b), and as long as the trajectory stays within $S^b$ it continually moves through lower level surfaces—except, perhaps, momentarily at points like $\alpha$ on $M^b$. If $0 \notin S^b$, the
trajectory will cross \( W \) at some point, as for example at \( \beta \). There the trajectory makes a sharp bend as the governing system switches and the trajectory comes under the influence of differential system (a). Once in \( S^a \), the trajectory continues to move through ever lower level surfaces. Depending upon the shape of the switching line \( W \), the trajectory may move back and forth between \( S^a \) and \( S^b \) several times but it must approach the origin asymptotically.\(^5\)

We now take this relatively simple theorem and show how it can be useful in the study of non-Walrasian dynamics.

3.

Before working any specific problems, it is worth our while to first prove a very useful theorem on the connection between Routh-Hurwitz conditions and Lyapunov functions.\(^6\)

Consider the two-dimensional linear autonomous system

\[
\begin{align*}
\dot{x} &= ax + by, \\
\dot{y} &= cx + dy,
\end{align*}
\]

where \( a, b, c, \) and \( d \) are constants.

\( (L) \) is of interest not only in the study of linear systems, but also in the study of local properties of nonlinear systems, where \( a, b, c, \) and \( d \) are taken to be various partial derivatives evaluated at the equilibrium. We wish to show the following:

**Theorem 3:** If the Routh-Hurwitz conditions \((a + d < 0 \text{ and } ad - bc > 0)\) hold for \((L)\), i.e., if \((L)\) is stable, then there is a function \( V(x, y) \) satisfying the conditions of Theorem 1.

**Proof:** We consider three cases: First, suppose \( b \neq 0 \). Try the function

\[
V(x, y) = (ax + by)^2 + (ad - bc)x^2.
\]

So

\[
\begin{align*}
\dot{V}(x, y) &= \dot{x}[2a^2x + 2aby + 2(ad - bc)x] + \dot{y}[2abx + 2b^2y] \\
&= (2a^2 + 2a^2d)x^2 + (4a^2b + 4abd)xy + (2ab^2 + 2b^2d)y^2 \\
&= 2(a + d)(ax + by)^2 \leq 0.
\end{align*}
\]

Let \( M = \{(x, y) | ax + by = \dot{x} = 0\} \). Then clearly, \( \dot{V}(x, y) < 0 \) for \((x, y) \not\in M\), and \( \dot{V}(x, y) = 0 \) for \((x, y) \in M\). We need only show now that \( M \) cannot contain an entire trajectory other than 0. A sample graph of \( M \) is shown in Figure 3. The slope of \( M \) is \(-a/b\). Since \( b \neq 0 \), \( M \) may slope any way except vertically. Pick any point on \( M \) other than 0, say \( \alpha \). Since \( \dot{x} = 0 \) at \( \alpha \), movement away from \( \alpha \) must either be straight up or straight down; hence, we must move off \( M \), and \( M \) can contain no entire trajectory other than 0.

\(^5\) Aoki has recently derived an interesting result similar to our Theorem 2 [1, p. 217], though his approach is quite different. We cannot discuss that work at length here, though we may note two important differences: first, in terms of methods, Aoki makes no use of Lyapunov functions; second, in results, Aoki's theorem is local only.

\(^6\) Such theorems are well known. See, for example, Hahn [7, p. 26].
Second, if \( b = 0 \) but \( c \neq 0 \), use the function \( V(x, y) = (cx + dy)^2 + (ad - bc)x^2 \) and proceed as above. And finally, if \( b = c = 0 \), use \( V(x, y) = x^2 + y^2 \). Then \( V(x, y) = 2ax^2 + 2dy^2 < 0 \) for all points off the origin, since the Routh-Hurwitz conditions imply \( a, d < 0 \).

With the help of Theorems 1–3, we now look at a sample non-Walrasian stability problem.

We are interested in capturing the central problem in non-Walrasian studies—spillovers. When a market fails to clear, we want rationed traders to take that fact into account when formulating demands later in the trading day. Once we take the spillover phenomenon seriously, a number of other considerations intrude almost at once: chiefly, learning, expectations, and monetary constraints. Some of these issues will be taken up in Section 5, but for the present we limit ourselves to a fairly simple sequential spillover model.

We are interested in the stability of an aggregative, three good economy with two flow goods, labor \( (q) \) and commodities \( (c) \), and a durable called “money.” There are two traders—the household and the firm. Trading takes place sequentially as follows: In the morning prices are posted, and traders proceed to the labor market. There the firm demands the profit maximizing quantity of labor \( (qd(P/P_c)) \) and the household offers \( q_s \), a constant. We suppose that \( Q, \) the lesser of \( q^d(\cdot) \) and \( q^s \), is actually traded. Traders then go to the commodity market, where the firm offers to sell \( F(Q) \)—all of the output produced by the actual quantity of labor hired. Note that \( F(Q) = F(q^S) \), if \( q^s \leq q^d(\cdot) \); and \( F(Q) = F(q^d(\cdot)) \), if \( q^d(\cdot) \leq q^s \).

Once out of the labor market the household may or may not have a second round decision to make. If the household finds itself unable to sell \( q^s \), it will be forced to revise its trading plans in light of its actual income, since it cannot hold consumption and ending money demands at their initial notional levels. It is reasonable to suppose that both constrained consumption demand, or “effective demand,” and ending money demand will be diminished by the inability to sell labor. Thus, denoting the effective demand by \( c^d(\cdot) \), we note that if at prices \( P^* \) and \( P_c^* \) we have \( Q = q^d(P^*_q/P^*_c) < q^s \), \( c^d(P^*_q, P^*_c, Q) < c^d(P^*_q, P^*_c) \). If on the other hand things went according to plan for the household in the labor market, i.e., if
Q = \bar{q}^s \leq q^d(\cdot), \text{ then the household's notional and effective demands will be the same. Thus, if } q^d(\hat{P}_q/\hat{P}_c) \geq \bar{q}^s = Q, \text{ then } c^d(\hat{P}_q, \hat{P}_c, Q) = c^d(\hat{P}_q, \hat{P}_c). \text{ In any case, the actual demand expressed in the commodity market is } \min (c^d(\cdot), c^d(\cdot)).

Note that the effect of a price change on c^d flows through two channels. First, there is the usual sort of price effect holding Q constant. And second, there is the indirect effect working through the influence of prices on Q and then Q on c^d. Using subscripts to denote partial differentiation and noting that Q = q^d(\cdot) whenever c^d is in effect, we note this by writing

\[ dc^d = c_{c^d}^d dP_c + c_{c^d}^d q^d dP_q + c_{c^d}^d q^d dP_q + c_{c^d}^d q^d dP_a. \]

In the Appendix we show that c_{c^d}^d = c_{c^d}^d and c_{c^d}^d = c_{c^d}^d, and this fact will be used subsequently.

We add the following assumptions.

**ASSUMPTION (C):** The functions q^d(\cdot), c^d(\cdot), and F(\cdot) are differentiable.

**ASSUMPTION (S):** (i) c^d_q, q^d_c, c_{q^d}^d, F'(\cdot) > 0 and c^d_q, q^d_c > 0; (ii) F'(\cdot) = P_q/P_c, if Q = q^d; (iii) c_{q^d}^d < P_q/P_c.

The first two parts of (S) are rather obvious, and the third is the analogue of the familiar Keynesian assumption that MPC < 1. Check this by writing (iii) as

\[ P_c dC^d/dP_q < 1 \]

and noting that the numerator is the value of the change in consumption and the denominator is the value of the change in income. Note that (ii) and (iii) imply that (c_{q^d}^d - F'(\cdot)) < 0; this will be of some use shortly.

Formally, we are interested in the stability of the system

(NW)

\[ \dot{P}_q = q^d(\cdot) - \bar{q}^s, \]
\[ \dot{P}_c = \min (c^d(\cdot), c^d(\cdot)) - F(\min (q^d(\cdot), \bar{q}^s)), \]

presuming for simplicity that the adjustment speeds are unity in both markets.\(^7\)

Since the non-Walrasian system (NW) is not everywhere differentiable, it is not amenable to the usual forms of stability analysis. It is however suited to the use of Theorem 2.

We note that in the space \{(P_q, P_c) | q^s \leq q^d(\cdot)\} = s^+, P_q and P_c are driven by the system

(NW\(^+\))

\[ \dot{P}_q = q^d(\cdot) - \bar{q}^s, \]
\[ \dot{P}_c = c^d(\cdot) - F(\bar{q}^s). \]

And in \(s^- = \{(P_q, P_c) | q^s \geq q^d(\cdot)\}, \) we have

(NW\(^-\))

\[ \dot{P}_q = q^d(\cdot) - \bar{q}^s, \]
\[ \dot{P}_c = c^d(\cdot) - F(q^d(\cdot)). \]

\(^7\) Using the more general price adjustment rules \( \dot{P}_q = H_q[q^d(\cdot) - \bar{q}^s] \) and \( \dot{P}_c = H_c[\min (c^d(\cdot), c^d(\cdot)) - F(\min (q^d(\cdot), \bar{q}^s))], \) where \( H_q(0) = 0 \) and \( H_q' > 0, \) has no effect upon the following other than complicating notation.
Before discussing the stability of (NW) we establish a preliminary result (Lemma 1) and make the following assumption.

**ASSUMPTION (W):** *The Walrasian system corresponding to (NW), i.e., the system*

\[
\begin{align*}
\dot{P}_q &= q^d(\cdot) - \bar{q}^s, \\
\dot{P}_c &= c^d(\cdot) - F(q^d(\cdot)),
\end{align*}
\]

is locally stable about a unique equilibrium point. This implies that \(q_a^d(c_c^d - F'(\cdot)q_c^d) - q_c^d(c_a^d - F'(\cdot)q_a^d) = q_a^d c_c^d - q_c^d c_a^d > 0\), where the partials are evaluated at the equilibrium.

Since we are interested in the destabilizing influence (if any) of spillovers, Assumption (W) is quite natural. If (NW) fails to be stable given Assumption (W), we can then see clearly that the source of the instability lies in the spillover phenomenon.

**LEMMA 1:** *Systems (W), (NW'), and (NW-) have a common equilibrium.*

**PROOF:** Suppose \((P_{q^*}, P_{c^*})\) gives \(q^d(P_{q^*}/P_{c^*}) = \bar{q}^s\), and \(c^d(P_{q^*}, P_{c^*}) = F(q^d(P_{q^*}/P_{c^*}))\), then \(\dot{P}_q = 0\) for (W), (NW'), and (NW-) and \(\dot{P}_c = 0\) for (W). Since \(q^d(\cdot) = \bar{q}^s\), \(F(q^d(\cdot)) = F(\bar{q}^s)\), and therefore \(c^d(P_{q^*}, P_{c^*}) = F(\bar{q}^s)\) and \(\dot{P}_c = 0\) for (NW'). Also, since \(Q = \bar{q}^s\), \(q^d(\cdot) = c^d(\cdot)\), and therefore \(c^d(P_{q^*}, P_{c^*}, Q) = F(q^d(P_{q^*}/P_{c^*}))\), so \(\dot{P}_c = 0\) for (NW'). Thus, \((P_{q^*}, P_{c^*})\) is an equilibrium for all three systems.

It is now easy to prove the following theorem.

**THEOREM 4:** *System (NW) is locally stable.*

**PROOF:** We have shown that (NW') and (NW-) have the same equilibrium, so we need now only show that there is some function \(V(P_q, P_c)\) which is decreasing away from the origin along all trajectories in \(s^+\) and \(s^-\). With Theorem 3 in mind, we start by checking the Jacobians. The Jacobian of (NW') is

\[
J^{NW'} = \begin{pmatrix} q_a^d & q_a^d \\ c_q^d & c_c^d \end{pmatrix}.
\]

Clearly, \(\text{tr} (J^{NW'}) < 0\). And further \(|J^{NW'}| = q_a^d c_c^d - q_c^d c_a^d > 0\), by Assumption (W). This establishes that \(V(P_q, P_c) = (q_a^d P_q + q_c^d P_c)^2 + (q_a^d c_q^d - q_c^d c_a^d)P_q^2\) is decreasing along all trajectories in \(s^+\). Using \(c_c^d = c_c^d\) and \(c_a^d = c_a^d\) as shown in the Appendix, the Jacobian for (NW') can be written

\[
J^{NW'} = \begin{pmatrix} q_a^d & q_a^d \\ c_q^d + c_c^d q_a^d - F'(\cdot)q_a^d & c_c^d + c_c^d q_a^d - F'(\cdot)q_c^d \end{pmatrix}.
\]
By Assumption (S), \( \text{tr} (J^\text{NW}) = q^d + c^d + q^c (c^d - F'(c^d)) < 0. \) And \( |J^\text{NW}| = q^a c^d - q^c c^d > 0, \) again by Assumption (W). So the same function \( V(P_q, P_c) \) used above is also decreasing along trajectories in \( s^- \) as \( (P_q, P_c) \) is driven by \( (NW^-) \). Thus \( (NW) \) satisfies the requirements of Theorem 2 and is locally stable.

4.

The results of the previous section should be extended along a number of lines. Mathematically, we would like to be able to give up the assumption of isolated equilibria. Economically, we should consider systems which not only have non-Walrasian processes, but also have non-Walrasian equilibria, and we are anxious to introduce expectations into the spillover process, since the mistakes which traders make in disequilibrium\(^8\) would naturally prompt them to form expectations. In the present section we simultaneously pursue all of these issues. We begin by deriving the main theorem for this part.

We start with the following definition.

**Definition:** The set \( A \) is called strongly attractive if (i) \( x(t) \in A \) implies \( x(T) \in A \) for all \( T > t \). (ii) For any \( x(0) \in A \), \( \lim_{t \to \infty} x(t|x(0)) \in E \), where \( E \) is the equilibrium set. (iii) If \( x(0) \notin A \), then either there is some finite \( t' \) such that \( x(t'|x(0)) \in A \), or \( \lim_{t \to \infty} x(t|x(0)) \in E. \)

In plain language, a set is strongly attractive if (i) once a trajectory enters \( A \), it cannot escape, (ii) all trajectories emanating from points in \( A \) converge to \( E \), and (iii) all trajectories emanating from points out of \( A \) are either carried into \( A \), thence into \( E \), or approach directly some point in \( E \) on the border of \( A \).

This leads easily to the following theorem.

**Theorem 5:** If a system has a strongly attractive set, then it is quasi-globally stable.

**Proof:** Since we know that for all \( x(0) \in A \), \( \lim_{t \to \infty} x(t|x(0)) \in E \), we need only show that the same holds for \( x(0) \notin A \). But this follows easily since, by (iii) of the definition, either there must be some \( t' \) such that \( x(t'|x(0)) \in A \), and then (ii) insures that \( \lim_{t \to \infty} x(t|x(t')) = \lim_{t \to \infty} x(t|x(0)) \in E, \) or simply \( \lim_{t \to \infty} x(t|x(0)) \in E. \)

Something like our Theorem 5 has been appealed to implicitly for some time in economic stability arguments. Figure 4 shows a well known case.\(^9\) Let \( A \) be the shaded area including the borders \( P_1 = 0 \) and \( P_2 = 0. \) Clearly, trajectories emanating from points in \( A \) (i) cannot escape, and (ii) must approach \( E. \) Also, tra-

\(^8\) In the previous model the firm would sometimes hire labor and then find that it could not sell the output. Had the firm known that it was to be sales constrained, it would have been able to make greater profits by hiring less labor.

\(^9\) See Arrow and Hurwicz [3, pp. 547 ff].
jectories emanating from points to the right (left) of $A$ have a leftward (rightward) component of motion and must either approach $E$ directly or move into $A$. We know that they cannot approach some non-equilibrium point on the border of $A$, since under the usual assumptions every limit point will be an equilibrium.

5.

To illustrate the piecewise use of Theorem 5, we will consider a model similar in some ways to one developed by Varian in a recent paper. We construct a two good, two trader, pure flow production economy, which has the firm hiring labor on the basis of point commodity sales expectations, with the household subject to income constraints in the commodity market. Formally, we are interested in the stability of

$$\dot{w} = g(\min(q^d(w), q^d(y)) - q^s(w)),$$

$$\dot{y} = h(F(\min(q^d(w), q^d(y), q^s(w))) - y),$$

where the notation is the same as in Varian and the earlier part of this paper; $w$ is the real wage, and $y$ is the expected commodity demand.

We can think of the system operating as follows: $w$ and $y$ are announced at the start of the period. The firm then computes its notional demand for labor, $q^d(w)$, together with $F(q^d(w))$. If $F(q^d(w)) > y$, the firm limits its labor demand to $q^d(y) = F^{-1}(y)$. So actual labor demand will be $\min(q^d(w), q^d(y))$. At the same time the household determines $q^s(w)$, a notional quantity.\(^{10}\) The firm and the household then meet in the labor market, where the quantity $\min(q^d(w), q^d(y), q^s(w)) = Q(w, y)$ is contracted, and where the actual excess demand is measured for the purpose of setting $w$.

\(^{10}\) Ideally we should also have the household’s labor supply partly dependent upon expectations about commodity supply, so it can try to avoid situations wherein it gives up leisure only to find that because of a shortage in the commodity market, it cannot buy the desired quantity of goods. As we shall see, such considerations are unnecessary in this particular model—and luckily so, since a third dynamic variable would complicate matters rather seriously.
We suppose that all realized profits are paid to the household and that the commodity is the only use of funds for the household.\textsuperscript{11} Thus, commodity demand is \( F(Q(w, y)) \).\textsuperscript{12} The firm then adjusts expectations according to the difference between actual and expected commodity demand.

We make the following assumptions:

**Assumption (C'):** The functions \( q^d(w) \), \( q^d(y) \), \( q^s(w) \), and \( F(\cdot) \) are all differentiable.

**Assumption (S'):** \( q^d_y(y), q^s_w(w), F'(\cdot) > 0 \), \( q^d_w(w) < 0 \).

As in Varian’s system, \( I \) has two sorts of equilibria: Walrasian equilibrium at which \( q^d(w) = q^d(y) = q^s(w) \), and non-Walrasian equilibria at which \( q^d(w) > q^d(y) = q^s(w) \). In contrast with Varian, however, the non-Walrasian equilibrium points form a connected set, while the Walrasian equilibrium is a limit point on the set of non-Walrasian equilibria.\textsuperscript{13} Denote \( E = \{(w, y) | q^d(w) \geq q^d(y) = q^s(w)\} \); then \( E \) is the equilibrium set.

A sketch of the graph of \( E \) is shown in Figure 5. \((w^*, y^*)\) is the Walrasian equilibrium, and the points on \( E \) to the southwest are non-Walrasian equilibria.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5}
\caption{Figure 5}
\end{figure}

\textsuperscript{11} In Varian’s analysis there is a sort of “phantom money,” which can serve as both a source and use of funds for the household. We have not followed him on this point, because it does not seem satisfactory to introduce such a commodity and then not take it seriously in the whole of the analysis. Varian’s “money” does not limit the value of purchases, nor does it change hands with each transaction or generate wealth effects. Also, since our system has no “money,” we cannot suppose as Varian does that profits or consumption out of profits are fixed at the Walrasian level. This difference in construction is responsible for the fact that Varian’s system has isolated equilibria, while ours has a connected equilibrium set. Some discussion of this is given by Varian [10, p. 577].

\textsuperscript{12} Proof: Let \( R \) be actual real profits, and let \( Y(w, y) \) be actual commodity demand. Then \( Y(w, y) = R + wQ(w, y) \). But \( R = F(Q(w, y)) - w \cdot Q(w, y) \). Thus \( Y(w, y) = F(Q(w, y)) \). See Varian [10, pp. 576–577].

\textsuperscript{13} See note 11 for a brief explanation of this difference.
We know that $E$ is upward sloping, since its slope is $q^d/q^s_w$. And we know that $E$ does not rise above $w^*$, since we would then have $q^s(w) > q^d(w)$ from (S'); nor does $E$ lie in the half-plane $y > y^*$, since we would then need $w > w^*$ to get $q^d(y) = q^s(w)$.

We now show the following theorem.

**Theorem 6:** (I) contains a strongly attractive set and thus is quasiglobally stable.

**Proof:** Consider the set $A = \{(w, y) | w \leq w^*\}$, where $(w^*, y^*)$ is the Walrasian equilibrium. We show that $A$ is strongly attractive.

(i) By checking the slopes of the trajectories along the border line $(w = w^*)$ of $A$, we can see that once in $A$, a trajectory cannot escape $A$. (a) Along the segment $\{(w, y) | w = w^*; y < y^*\}$ using (S') we have $q^d(y) < q^d(y^*) = q^d(w^*) = q^s(w^*)$. Thus $\dot{w} < 0$. (b) At $(w^*, y^*)$ we clearly have $\dot{w} = 0$. (c) And along $\{(w, y) | w = w^*; y > y^*\}$ we have $q^s(w^*) = q^d(w^*) = q^d(y^*) < q^d(y)$, hence $\dot{w} = 0$.

Note also that in this case $Q(\cdot) = q^d(w) = q^s(w) < q^d(y)$. Thus $F(Q(\cdot)) < F(q^d(y)) = y$, and therefore $\dot{y} < 0$. This makes $\{(w, y) | w = w^*; y > y^*\} = L$ a "sliding line" in the language of Barbashin.\(^{14}\) Once a trajectory is carried to $L$, it slides to the Walrasian equilibrium. This invests $(w^*, y^*)$ with a stability property not shared by the non-Walrasian equilibria. Since $w$ cannot increase along the border $w = w^*$, we know that once in $A$ a trajectory cannot escape.

(ii) To simplify notation, we define the vector $\chi = (w, y)$. To show that $\lim_{t \to \infty} \chi(t) | \chi(0) \in E$ for all $\chi(0) \in A$, it is sufficient to show that there is some continuous function $V(\chi)$ defined on $A$ such that (a) $V(\chi) \geq 0$, (b) $V(\chi) = 0$ iff $\chi \in E$, and (c) $V(\chi)$ is decreasing along all trajectories in $A$ outside $E$.

As a preliminary we define the functions $\dot{y}(w(t))$ and $\dot{w}(y(t))$. Given any $w \leq w^*$, we know that there is some level of $y$, call it $\tilde{y}(w(t))$, such that $(w(t), \tilde{y}(w(t))) \in E$. $\tilde{y}(w(t))$ is the value of $y$ which gives $q^d(\tilde{y}(w(t))) = q^s(w(t))$ for some particular value of $w$. Note that $\tilde{y}(w(t))$ is a differentiable function and that $d\tilde{y}/dw > 0$. Similarly, we can define $\tilde{w}(y(t))$ by $(\tilde{w}(y(t)), y(t)) \in E$ for $y \leq y^*$ and let $\tilde{w}(y(t)) = w^*$ for $y > y^*$. $\tilde{w}(y(t))$ is also a continuous function, differentiable everywhere but at $y^*$, with $d\tilde{w}/dy > 0$ for $y < y^*$, and $d\tilde{w}/dy = 0$ for $y > y^*$.

We now show that the continuous function $V(t) = |w(t) - \tilde{w}(y(t))|$ + $|y(t) - \tilde{y}(w(t))|$ on $A$ satisfies conditions (a)--(c).

(a) Obviously $V(t) \geq 0$.

(b) $V(\chi) = 0$ iff $\chi \in E$. It is easy to show that $(\tilde{w}, \tilde{y}) \in E \implies V(\tilde{w}, \tilde{y}) = 0$, for by the definition of $\tilde{y}(w(t))$ and $\tilde{w}(y(t))$ at any equilibrium we have $\tilde{w}(t) = \tilde{w}(\tilde{y}(t))$ and $\tilde{y}(t) = \tilde{y}(\tilde{w}(t))$. Hence, $V(\tilde{w}, \tilde{y}) = 0$. To show $V(\tilde{w}, \tilde{y}) = 0 \implies (\tilde{w}, \tilde{y}) \in E$, we note that if $V(\chi) = 0$, it must be that $\tilde{y}(t) = \tilde{y}(\tilde{w}(t))$, but then $(\tilde{w}, \tilde{y}) \in E$ by the definition of $\tilde{y}(\cdot)$.

(c) To show that $V(\chi)$ is everywhere decreasing along trajectories in $A \setminus E$, we use

\(^{14}\) Barbashin [4, p. 122].
partition $A \setminus E$ into four sectors and check $V(t)$ in each sector. Define the sectors

- $S_1 = \{(w, y) | w = w^*; y \geq y^* \}$
- $S_2 = \{(w, y) | w < w^*; y \geq y^* \}$
- $S_3 = \{(w, y) | w < w^*; y < y^*; q^d(y) > q^s(w) \}$
- $S_4 = \{(w, y) | w < w^*; y < y^*; q^d(y) < q^s(w) \}$

Thus using the definitions of $\tilde{w}(\cdot)$ and $\tilde{y}(\cdot)$ together with the properties of $E$, we have the following corresponding values of $V(t)$.\(^{15}\)

\[
V(t) = \begin{cases} 
  y(t) - y^*, & \chi \in S_1, \\
  w^* - w(t) + y(t) - \tilde{y}(w(t)), & \chi \in S_2, \\
  \tilde{w}(y(t)) - w(t) + y(t) - \tilde{y}(w(t)), & \chi \in S_3, \\
  w(t) - \tilde{w}(y(t)) + \tilde{y}(w(t)) - y(t), & \chi \in S_4.
\end{cases}
\]

- $\chi \in S_1$: In $S_1$, $\dot{y} < 0$, since $F(Q) = F(q^d(w^*)) < F(q^d(y)) = y$. Then since $\dot{V} = \dot{y}$, we clearly have $\dot{V} < 0$.
- $\chi \in S_2$: Since $w < w^*$, we have $q^d(w) > q^d(w^*) = q^s(w^*) > q^s(w)$. And since $y \geq y^*$, we have $q^d(y) \geq q^d(y^*)$. Since $q^d(y^*) = q^s(w^*), \min(q^d(w), q^d(y)) > q^s(w)$, and, therefore, $\dot{w} > 0$. Also, since $Q(\cdot) = q^s(w) < q^d(y)$, $F(Q(\cdot)) < F(q^d(y)) = y$, and $\dot{y} < 0$. Now in $S_2$, $\dot{V}(\cdot) = -\dot{w} + \dot{y} - (\frac{d\tilde{y}}{d\tilde{w}})\dot{w} < 0$, since $\frac{d\tilde{y}}{d\tilde{w}} > 0$.
- $\chi \in S_3$: With $w < w^*$ we have $q^d(w) > q^d(w^*) = q^s(w^*) > q^s(w)$. And in $S_3$ we have $q^d(y) > q^s(y)$, therefore $\min(q^d(y), q^d(w)) > q^s(w)$. So $\dot{w} > 0$. Also, since $Q = q^s(w) < q^d(y)$, $F(Q) < y$, and $\dot{y} < 0$. So in $S_3$, $\dot{V}(\cdot) = (\frac{d\tilde{w}}{d\tilde{y}})\dot{y} - \dot{w} + \dot{y} - (\frac{d\tilde{y}}{d\tilde{w}})\dot{w} < 0$, since both $\frac{d\tilde{w}}{d\tilde{y}}$ and $\frac{d\tilde{y}}{d\tilde{w}}$ are positive.
- $\chi \in S_4$: Again we have $q^d(w) > q^s(w)$, but now $q^d(y) < q^s(y)$, thus $\dot{w} < 0$. Now $Q = q^d(y)$, so $F(Q) = F(q^d(y)) = y$, and $\dot{y} = 0$. Thus $\dot{V}(\cdot) = \dot{w} - (\frac{d\tilde{w}}{d\tilde{y}})\dot{y} + (\frac{d\tilde{y}}{d\tilde{w}})\dot{w} - \dot{y} < 0$.

This establishes that once in $A$, all trajectories must be carried arbitrarily close to $E$. To show that $A$ is strongly attractive, only the following remains to be shown:

(iii) Points outside $A$ either move into $A$ in finite time or approach a point in $E$ on the border of $A$. We show this by first noting that if $w > w^*$, $\dot{w} < 0$.\(^{16}\) Now if $w$ is decreasing everywhere above $w^*$, and if the path of $(w(t), y(t))$ is bounded,\(^{17}\) then clearly there must either be some finite $T$ such that $(w(T), y(T)) = (w^*, y(T))$ or

\(^{15}\)Take points in $S^2$ for example. By the definition of $\tilde{w}, y \geq y^*$ implies $\tilde{w} = w^*$. Then since $w < w^*, w < \tilde{w}$ and clearly $|w - \tilde{w}| = w - w$. And since $E$ lies in $\{(w, y) | w < w^*; y \leq y^*\}$, using the definition of $\tilde{y}$, we know that if $y \geq y^*$, then $y \geq \tilde{y}$. Thus $|y - \tilde{y}| = y - \tilde{y}$. So $V = |w - \tilde{w}| + |y - \tilde{y}| = \tilde{w} - w + y - \tilde{y}$.

\(^{16}\)PROOF: $w > w^*$ and Assumption $(S')$ imply $q^d(w) < q^d(w^*) = q^s(w^*) < q^s(w)$, and $q^d(w) < q^s(w)$. But then $\min(q^d(w), q^d(y)) < q^s(w)$, so $\dot{w} < 0$.

\(^{17}\)This is easily shown: Since $w > w^*$ implies $\dot{w} < 0$, $w(t)$ is clearly bounded above. Also, we can show that $y < y^*$ implies $\dot{y} < 0$, so $y(t)$ is bounded above. Then taking $w = 0$ and $y = 0$ as the lower bounds, we have established that $\chi(t)(\chi(0))$ is bounded.
there is some subsequence \( \chi(t_k) \), such that \( \lim_{k \to \infty} (w(t_k), y(t_k)) = (w^*, \hat{y}) \) for some \( \hat{y} \). That is, \( w(t) \) must either (a) cross through some point on the line \( w = w^* \) or (b) approach arbitrarily close to some point(s) on that line. If (a), i.e., if \( \chi \) enters \( A \), we know from (i) and (ii) of this theorem that the trajectory will approach \( E \). And if (b), that is, if \( \chi(t) \) approaches some point, say \( \hat{x} \), on \( w = w^* \), we can be sure that \( \hat{x} \in E \).\(^{18}\)

(i), (ii), and (iii) establish that \( A \) is strongly attractive, and this in turn proves that (I) is quasi-globally stable.

Though our purpose here has been more in the nature of displaying tools than arguing economics per se, it is worth pointing out that some fairly subtle differences between Varian’s model and ours have produced rather dramatically different stability properties. Varian’s system was shown to have “one-sided instability at the Walras equilibrium, and stability at the non-Walrasian equilibrium,”\(^{19}\) while our equilibrium set, comprising both Walrasian and non-Walrasian equilibria, is quasi-globally stable, and the Walrasian equilibrium in fact has a sliding line.\(^{20}\) The lesson must be that models of this sort are rather sensitive to their specification.

6.

If the study of non-Walrasian processes is to go forward, it is clear that new stability tools will have to be employed. In the present paper we derived and illustrated the use of two relatively simple extensions of Lyapunov’s classic theorem. No doubt stronger and more useful theorems can be found, but already our results have taken us further toward an understanding of non-Walrasian system. The economic models used can be taken as nothing more than crude first approximations, and much remains to be done in this regard. It seems that a minimally acceptable non-Walrasian model would have three goods, one of which serves as a true medium of exchange (as opposed to simply being named “\( m \)),” two traders, spillovers, and expectations. We still seem to be a long way from that, but given the recent advances by Varian, Fisher, Aoki, and others, the end may be in sight. It is hoped that the methods of the present paper will help achieve that end.

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\(^{18}\) PROOF: We show that \( \chi \in E \) implies either \( \dot{w} < 0 \) or \( \dot{y} < 0 \); then an easy modification of a theorem by Arrow and Hahn \cite{2} will establish the desired result. Suppose the contrary, i.e., suppose \( \chi \in E \) and \( \dot{w} \geq 0 \) and \( \dot{y} \geq 0 \). If \( \dot{w} \geq 0 \), then \( \min(q^d(w), q^d(y)) \geq q^*(w) \) and certainly \( q^d(y) \geq q^*(w) \). Also, \( Q = q^*(w) \). If, then \( \dot{y} \geq 0 \), \( F(q^*(w)) \geq y = F(q^*(y)) \). But then \( q^*(w) \geq q^*(y) \), so \( q^*(w) = q^*(y) \). Thus we have \( q^d(w) = q^d(y) = q^d(y) \), so \( \chi \in E \). But this is a contradiction. Thus \( \chi \in E \) implies either \( \dot{w} < 0 \) or \( \dot{y} < 0 \). Use this in place of the second line in the proof of Arrow and Hahn’s Theorem 11.2, and the desired result follows easily, i.e., limit points are shown to be equilibria.

\(^{19}\) Varian \cite[p. 585]{10}.

\(^{20}\) This is not to say that (I) has a strong tendency to the Walrasian equilibrium. A look at the phase portrait shows, for example, that any perturbation to a “sub-Walrasian” level of expectations must lead ultimately to a non-Walrasian equilibrium.
APPENDIX

We show that $c_i^{d'} = c_i^{d_d}, i = c, q$, when evaluated at the equilibrium.

Since the quantity of labor sold is determined exogenously from the point of view of the household, we can work with the utility function $U(c, m)$. The household’s problem is then to maximize $U(c, m)$ subject to $P_c Q + \bar{m} - P_c c - m = 0$, where $\bar{m}(m)$ is the initial (desired ending) money holding. Consider first changes in $P_c$ and $P_q$ leading into $s^+$, so that $Q = \bar{Q}^+$ and $dQ/dP_c = dQ/dP_q = 0$. Totally differentiating the first order conditions we have

$$
\begin{bmatrix}
U_{cc} & U_{cm} & -P_c \\
U_{mc} & U_{mm} & -1 \\
-P_c & -1 & 0
\end{bmatrix}
\begin{bmatrix}
dc \\
dm \\
d\lambda
\end{bmatrix}
= \begin{bmatrix}
\lambda dP_c \\
0 \\
-P_q dQ - QdP_q - d\bar{m} + cdP_c
\end{bmatrix}.
$$

Letting $D$ be the bordered Hessian above, we have

$$
dc = \begin{bmatrix}
\lambda dP_c & U_{cm} & -P_c \\
0 & U_{mm} & -1 \\
-P_q dQ - QdP_q - d\bar{m} + cdP_c & -1 & 0
\end{bmatrix} \cdot \frac{1}{|D|}.
$$

Thus

$$
\frac{\partial c^d}{\partial P_c} = c^d_c = (U_{cm} c - \lambda + P_c U_{mm} c) \frac{1}{|D|}
$$

and

$$
\frac{\partial c^d}{\partial P_q} = c^d_q = (U_{cm} Q - P_c U_{mm} Q) \frac{1}{|D|}.
$$

Now consider the one-sided derivatives leading into $s^-$, now $Q = q(d^d)$, and $Q$ can change with $P_c$ and $P_q$. The household’s problem is still to maximize $U(c, m)$ subject to $P_q Q + \bar{m} - P_c c - m = 0$, but now $dQ = q(d^d) P_q + q(d^d) P_c$. This yields

$$
dc^d = \begin{bmatrix}
\lambda dP_c & U_{cm} & -P_c \\
0 & U_{mm} & -1 \\
-QdP_q - P_q dQ dP_q - P_q dQ dP_c - d\bar{m} + cdP_c & -1 & 0
\end{bmatrix} \cdot \frac{1}{|D|}.
$$

Noting the values of $c^d_c$ and $c^d_q$ from above, we now have

$$(A^1) \quad \frac{dc^d}{dP_c} = (U_{cm} c + U_{cm} P_q q^d c - \lambda - P_c U_{mm} P_q q^d c + P_c U_{mm} c) \frac{1}{|D|}$$

$$
= c^d_c + \frac{1}{|D|} (U_{cm} P_q - P_c P_q U_{mm}) q^d c
$$

and

$$(A^2) \quad \frac{dc^d}{dP_q} = (U_{cm} Q + U_{cm} P_q q^d Q - P_c Q U_{mm} - P_c P_q q^d U_{mm}) \frac{1}{|D|}$$

$$
= c^d_q + \frac{1}{|D|} (U_{cm} P_q - P_c P_q U_{mm}) q^d q
$$

where all derivatives are assumed to be evaluated at the equilibrium $(Q, C)$ level. Noting that

$$
\frac{dc^d}{dQ} = c^d'_c = \frac{1}{|D|} (U_{cm} P_q - P_c P_q U_{mm}), \quad \text{and}
$$

$$
\frac{dc^d}{dP_c} = c^d_c + c^d'_d P_q,
$$

$$
\frac{dc^d}{dP_q} = c^d_q + c^d'_d Q,
$$

we see at once by substitution into $(A^1)$ and $(A^2)$ that $C^d_c = C^d_c$ and $C^d_q = C^d_q$. 
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