Closed-Form Solutions to Bundling Problems

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This paper investigates the case of a monopolist selling two distinct goods to a group of m traders who are characterized by their reservation prices, which are drawn independently from uniform distributions over the intervals \([0, R_1]\) and \([0, R_2]\). Closed-form solutions are derived for optimal prices, quantities, profits, and consumers’ surplus under situations of separate sales, pure bundling, and mixed bundling. This allows a clear comparison of the price, output, and welfare effects of various forms of bundling. We further investigate situations of positive marginal cost, positive and negative correlation of reservation values, and substitutes and complements.

1. Introduction

A monopolist who offers a single good at the same price to all potential buyers is likely to be frustrated by two facts: First, the seller knows that there are buyers who would have been willing to pay more than the current price. Second, the seller knows that some of the potential buyers who choose not to buy the product would have been willing to pay a price below the monopolist’s asking price but above the marginal cost. This “single-price frustration” has been studied for years under the heading of price discrimination, that is, charging different prices to different buyers of the same product. Multiproduct firms have another option which can relieve the single-price frustration: they can bundle their products and sell them as a single item, such as a computer along with an office suite, or a car with a navigation system. Bundling may either be “pure,” meaning only the bundle is offered, or it may be “mixed,” meaning that both the bundle and the separate goods are offered.

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Examples of bundling abound. In today’s paper, Home Depot is offering a Ryobi power screwdriver bundled with “free” electric snips, and American Eagle Outfitters offers half-off for all second pairs of pants with every regular price purchase. Is there any science to these specific pricing schemes? Are there well-defined and plausible optimization problems that yield these particular solutions? The answer, as we will see, turns out to be yes. And understanding that is one of the big payoffs to having closed-form results.

The literature on the economics of bundling dates back to at least Stigler (1963) and began to grow rapidly after Adams and Yellen (1976). Presently, there are at least two main threads to this literature. One thread deals with bundling as a strategy for firms in a multi-firm environment. Schmalensee’s 1982 paper is seminal in this literature. Two more recent examples are Nalebuff (2004) and Thanassoulis (2007). The second thread, to which this paper belongs, involves the use of bundling as a profit maximizing technique for an isolated monopolist. Both the Stigler and the Adams and Yellen papers were of this second type, and both of these papers worked with graphics and discrete numerical examples, finding that bundling can raise profits when reservation values are negatively correlated. Schmalensee (1984) explored a two-good model where traders’ reservation values derive from a bivariate normal distribution. No closed-form results were derived, but a series of numerical simulations led to “suggestive results.” Schmalensee speculated that bundling lowers consumers’ surplus and raises profits (when costs are low) even when reservation values are not negatively correlated. McAfee et al. (1989) found sufficient conditions for mixed bundling to dominate separate sales. Salinger (1995) explored a largely graphical method for investigating pure bundling with independent linear demands. Although there is a problem with his “aggregate components” benchmark to be discussed later, Salinger provided a strongly intuitive picture of why it is that bundling might raise consumers’ surplus and why bundling will cease to dominate separate sales when marginal costs are high. Bakos and Brynjolfsson (2001) broke new ground by examining the benefits of bundling a large number of goods. Their primary finding is that bundling many goods can result in the extraction by the seller of nearly all of the consumers’ surplus, although again high cost can cause separate sales to dominate. More recently, Chu et al. (2008) have written a fascinating empirical paper around the insight that firms bundling even a modest number of goods, $n$, face a very large number $(2^n - 1)$ of potential bundles. The authors propose that such firms offer $n$ bundles, each simply containing either 1 or 2 or … $n$ goods. Their example is a theater company with an eight-play season offering buyers a bundle of one play, two plays, etc.
Although no closed-form solutions are found, numerical experiments under a wide variety of assumptions lead the authors to believe that such a pricing strategy will attain about 99% of the profit from perfect mixed bundling. Two other recent empirical papers of interest come from Shiller and Waldfogel (2008), who explore pricing and revenue distribution options in the market for bundled music downloads, and Yurukoglu (2008) who explores the potential welfare implications of the unbundling of television channels. Coming closer to the thrust of this paper, Venkatesh and Kamakura (2003) studied the relative advantages of separate sales, pure bundling, and mixed bundling when two goods are complements, substitutes, or independent. Key assumptions are that the market-wide maximum reservations prices of the two goods, $R_1$ and $R_2$, are equal and that the distribution of buyers’ reservation prices is uniform. And the main finding is that optimal bundle prices rise (fall) as the goods are regarded as stronger complements (substitutes). Closed-form solutions are offered for the cases of separate sales and pure bundling, whereas mixed bundling is explored with numerical simulations. Interested readers can see Kobayashi (2005) for a much more extensive literature survey.

Most of the existing single-seller results can be characterized as either graphical/qualitative or what might be called “bounded conjecture.” Schmalensee (1984, p. S229) offers an example of what I mean by bounded conjecture when he writes, “In the symmetric case, mixed bundling is always more profitable than unbundled sales if $\rho \leq 0$ or if $\bar{\alpha}$ is large enough, and it is apparently more profitable for positive $\rho \neq 1$ at least as long as $\bar{\alpha} > -2.8$.” General theorems and closed-form solutions are rare, although they do exist. For example, Salinger is able to show that the optimal pure bundling price for two goods with identical maximum reservation values is $\sqrt{2/3}R$, where $R$ is the maximum reservation value for each of the goods. But such results tend to be found only under the simplest of conditions, such as $R_1 = R_2$, zero marginal cost, or pure bundling only, and typically only a few variables of interest are found, like the pure bundle price, but not the aggregate consumers’ surplus or the mixed bundle optimum prices. Still there are exceptions, for example, both Bakos and Brynjolfsson and Venkatesh and Kamakura provide a number of general theorems.

In this paper, with the use of analytic geometry and a good deal of brute force, we are able to find closed-form solutions for prices, profits, and consumers’ surplus for pure and mixed bundling under a variety of circumstances. This should be regarded as a clear advance in the topic. It is one thing to be able to show that mixed bundling raises profits compared with pure bundling, but it is more valuable yet to be able to compute and compare the two profit levels, the two sets of prices,
and the two levels of consumers’ surpluses given our assumptions about the distribution of buyers’ reservation prices. Our closed-form solutions give rise to a number of very specific and novel results. For example, we find that in certain circumstances the optimal mixed bundling price is higher than the pure bundling price by \((1/12)R_1\), where \(R_1\) is the maximum reservation value of the more expensive good. Or we find that in certain cases pure bundling will not improve profits if marginal cost exceeds 19% of \(R_1\). Another innovation here is that we are able to shed considerable light on “asymmetric” cases where a good with a relatively high valuation (i.e., \(R_1\) is high) is bundled with a good of lower value (\(R_2\) is low). This case has been consistently overlooked in the interest of keeping the analysis tractable by reducing variables with the use of the assumption that \(R_1 = R_2 = R\). But we find that with a bit of extra complexity, many interesting and unexplored phenomena emerge when the maximum reservation values are allowed to differ. Indeed, a number of pricing strategies observed in the marketplace come into focus. We can show why it sometimes pays to offer the bundle containing goods 1 and 2, and to offer good 2 alone, but not to offer good 1 separately. This is a novel finding that a referee aptly termed “partial mixed bundling.” Or we can show why it might be optimal to let consumers have one good “for free” whenever they buy the other good at the “regular price.”

The organization of the paper is straightforward. After laying out the notation and key assumptions, we proceed to finding optimum prices, profits, and consumers’ surplus under separate sales, pure, and mixed bundling with zero marginal cost. The closed-form solutions allow clear comparisons between the various pricing options. We then introduce positive marginal cost and move the analysis further forward. In the end, we briefly discuss perfect positive and negative correlation of reservation values, complements, and substitutes.

2. Preliminaries

Although closed-form theorems can solve problems and clarify issues in highly useful ways, the derivation of closed-form results can be tedious and inelegant. This paper is an abridgment of a much longer paper, “Closed-Form Solutions to Bundling Problems Complete Version,” which is available at http://www.csuchico.edu/~jeckalbar/bundling/. Interested readers can look to that paper for more depth of argument, more breadth of topics, and for most of the proofs. This is especially true with respect to the later topics, involving the bundling of complements or substitutes, the role of nonuniform distributions, and
correlations of a trader’s reservation values between two goods. The complete paper will be referred to here as “Closed-form Complete.”

We initially model a monopolist selling two goods to a set of \( m \) traders. We assume that traders’ reservation prices for good 1 are drawn from a uniform distribution over the interval \([0, R_1]\), and their reservation prices for good 2 are independently drawn from \([0, R_2]\). Without loss of generality, we assume that \( R_2 \leq R_1 \). Much of what goes on in the following sections is best visualized in \((P_1, P_2)\) space. Any trader’s reservation price-pair is a point, such as \((r_1, r_2)\), in \([0, R_1] \times [0, R_2]\) (see Figure 1). Note that because traders are uniformly distributed throughout \([0, R_1] \times [0, R_2]\), any area representing, say, 10% of the subspace will contain approximately 10% of all traders’ reservation price points. At a price of \( P_1' \), demand for good 1 will be \( Q_1(P_1') = (R_1 - P_1')R_2m/(R_1R_2) = k(R_1 - P_1')R_2 \). For convenience, we have introduced the term \( k = m/(R_1R_2) \), where \( k \) is the density of traders per unit area of \([0, R_1] \times [0, R_2]\). For most purposes, it is easier to work in terms of \( k \), although occasionally thinking in terms of \( m \) is more efficient. In all but a few cases toward the end of the paper, we will use \( k \).

If the monopolist were to sell the goods separately, demands could be computed as follows: If the price for good 1 were set at \( P_1' \), all of the traders whose reservation price points lie to the right of \( P_1' \) will be buyers of good 1, so the quantity demanded will be given by \( Q_1(P_1') = (R_1 - P_1')R_2m/(R_1R_2) \). This follows from the fact that the area of the rectangle containing the buyers is given by \((R_1 - P_1')R_2\), and
\[(R_1 - P'_1)R_2 / (R_1 R_2)\] gives the fraction of all buyers who will wish to buy at \(P'_1\). Multiplying the latter by \(m\) will give the number of buyers. (A further note on notation: The notation used here is conventional in most respects except that a space between variables indicates multiplication. This is the notation used in Mathematica, and it is adopted to make it easier for other researchers to use Mathematica with the functions presented here. This paper is also posted at the web site mentioned earlier as a Mathematica notebook and can be used interactively by other researchers.)

Solving the \(Q_1(P_1)\) equation for \(P_1\), we get the inverse demand equation \(P_1(Q_1) = R_1 - Q_1 / (k R_2)\). Total revenue from the sale of \(Q_1\) will be given by \(TR_1(Q_1) = R_1 Q_1 - Q_1^2 / k R_2\), and marginal revenue by \(MR_1(Q_1) = R_1 - 2 Q_1 / (k R_2)\). Similar calculations are easily done for good 2.

We assume initially that both fixed and marginal costs are zero for each good, so if the monopolist sells the goods separately, the optimum quantity for each good is given by \(k (R_1 R_2) / 2\), and the optimum prices will be \(R_i / 2, i = 1, 2\). The firm’s profit on good 1 will be \(k R_1^2 R_2 / 4\). The consumers’ surplus for good 1 is \(k R_1^2 R_2 / 8\). Note that the firm does not have to think in terms of reservation values. If the seller works with linear demand curves, \(R_i\) is simply the vertical intercept of the curve.

As mentioned already, any monopolist with a down-going demand curve who sells his product at the same price, \(P^*\), to all customers is apt to be frustrated by two facts: (i) He is likely to suspect that some of the buyers would have willingly paid more than \(P^*\). (ii) Because \(P^*\) exceeds marginal cost, the monopolist knows that he is forgoing sales to potential customers who are willing and able to pay less than \(P^*\) but more than the marginal cost, and this is a missed potential source of profit. These two facts prompt firms to search for alternative pricing strategies.

In a sense, the ideal demand curve would be horizontal, with all buyers having the same reservation price. In this case, the monopolist charges the common reservation price, and there are no foregone profits of the two sorts discussed earlier. More generally, these foregone profits tend to be small when the demand curve is fairly flat at prices above \(P^*\) and steep at prices below \(P^*\). We will see that this is precisely what bundling can do.

### 3. Pure Bundling

Now suppose the firm elects to bundle the two goods and sell them only in pairs at a bundle price of \(P_b\). Our goal is to derive closed-form
solutions for prices, quantities, profits, and consumers’ surplus to compare these results with the values of these key variables under separate sales and later under mixed bundling. To that end, we begin with a careful derivation of the bundle demand curve.

Consider the line $r_2 = P_b - r_1$ passing through the reservation price point $(r_1, r_2)$ in Figure 2a. Assuming that a buyer’s reservation price for the bundle is $r_1 + r_2$, any trader whose reservation price point is on or above this line will choose to buy the bundle. (We will look at alternate assumptions late in the paper.) We also assume that it is too costly for a trader to buy the bundle speculatively and then resell.
one or both of the goods. The characterization of the size of the buyer set depends upon the value of $P_b$ in relation to $R_1$ and $R_2$. We will find that the bundle demand curve has three distinct ranges—one linear and two nonlinear. The intuition behind that result is apparent in Figure 2a. Consider small reductions in $P_b$ from the value shown. Initially, the area holding the buyers is an expanding triangle like $A_1$. But when $P_b$ falls below $R_1$, each successive small drop in $P_b$ (until $P_b = R_2$) adds a parallelogram of new buyers. Thereafter, the set of new buyers resides in a trapezoid. We will build up the demand curve case by case.

**Case 1.** Suppose initially that $R_1 + R_2 > P_b > R_1$. All of the traders wishing to buy the bundle have reservation price points in area $A_1$ of Figure 2a. The area of $A_1$ is $(1/2)(R_1 + R_2 - P_b)^2$ and the number of traders wishing to buy the bundle will be given by $kA_1$. Let $Q_{b1}$ be the quantity of the bundle demanded in this circumstance. Then $Q_{b1}(P_b) = kA_1 = (k/2)(R_1 + R_2 - P_b)^2$. Solving for $P_b$, we get the inverse demand and marginal revenue equation under case 1,

$$P_{b1}(Q_{b1}) = R_1 + R_2 - \sqrt{\frac{2Q_{b1}}{k}} \quad \text{and} \quad MR_{b1}(Q_{b1}) = R_1 + R_2 - \frac{3\sqrt{Q_{b1}}}{\sqrt{2k}}.$$  

It is important to note that the equations for $P_{b1}$, $Q_{b1}$, and $MR_{b1}$ are only applicable when $P_b$ is in the region where $R_1 + R_2 > P_b > R_1$, which implies that $Q_{b1} < kR_2^2/2$.

**Case 2.** Suppose $R_1 > P_b > R_2$. Traders now wishing to buy the bundle reside in $A_2$ in Figure 2b. The area of $A_2$ is $(R_1 - P_b)R_2 + R_2^2/2$, and the quantity of the bundle demanded in case 2, $Q_{b2}$, is $kA_2$, or

$$Q_{b2}(P_b) = k(R_1 - P_b)R_2 + \frac{kR_2^2}{2}.$$  

Solving for $P_b$, we get the inverse demand and marginal revenue functions under case 2 as follows:

$$P_{b2}(Q_{b2}) = R_1 + \frac{R_2}{2} - \frac{Q_{b2}}{k_{b2}} \quad \text{and} \quad MR_{b2}(Q_{b2}) = R_1 + \frac{R_2}{2} - \frac{2Q_{b2}}{k_{b2}}.$$  

Note that the inverse demand and marginal revenue functions are linear in case 2. This follows from the fact that in this region, a unit change in $P_b$ always causes a gain or loss of a parallelogram of area $R_2$ in $A_2$. Note also that these functions apply when $kR_2^2/2 < Q_{b2} < kR_2(R_1 - R_2/2)$. It seems that most researchers, in assuming that $R_1 = R_2$, have missed this region on the bundle demand curve. As we will see, this results in overlooking a number of interesting situations.
FIGURE 3. BUNDLE DEMAND AND MARGINAL REVENUE WITH $k = 5$, $R_1 = 20$, AND $R_2 = 10$

Case 3. Consider the final case, where $R_1 > R_2 > P_b$. Bundle buyers now reside in $A_3$ (not shown), the area of which is given by $R_1 R_2 - \frac{P_b^2}{2}$.

This yields

$$Q_{b3}(P_b) = kR_1 R_2 - \frac{kP_b^2}{2}, \quad P_{b3}(Q_{b3}) = \sqrt{\frac{2kR_1 R_2 - 2Q_{b3}}{k}},$$

and

$$MR_{b3}(Q_{b3}) = \frac{-3Q_{b3} + 2kR_1 R_2}{k\sqrt{-\frac{2Q_{b3}}{k} + 2R_1 R_2}}.$$

The range of $Q_b$ over which case 3 applies is $kR_2(R_1 - R_2/2) \leq Q_{b3} \leq kR_1 R_2$.

The general form for the area containing bundle buyers in all three groups, call it $A$, can be given by

$$A(P_b) = -\frac{P_b^2}{2} + R_1 R_2 + \frac{1}{2}[\text{Max}[0, P_b - R_1]]^2 + \frac{1}{2}[\text{Max}[0, P_b - R_2]]^2.$$

It is important to note that $A(P_b)$ is a continuous, monotone function with a continuous derivative. And because $Q_b(P_b) = kA(P_b)$, we know that a continuous inverse demand, $P_b(Q_b)$ also exists. A sketch of the graph of the inverse bundle demand $P_b(Q_b)$ and its companion $MR_b$ for $k = 5, R_1 = 20, \text{and } R_2 = 10$ is shown in Figure 3. (Several
“benchmark” values are noted.) Notice the steepness of the bundle demand curve at both high and low prices and the flatness of the curve at intermediate prices. The steepness is due to the fact that there are relatively few traders with very high (or very low) bundle reservation prices, because this would require coincident high (or low) reservation values for both goods. With independently drawn reservation values, the large mass of traders are clustered near the middle values of $P_b$. This results in a flattening of the bundle demand curve in its mid-section, and this ultimately explains why sales can be higher under bundling than with separate sales.

Notice that in Figure 3 the profit maximum is found where the linear section of marginal revenue, that is, $MR_{b2}$, hits the $Q_b$ axis. We will refer to this as a Type I optimum. Not all optima will be of Type I. If $R_2 > (2/3)R_1$, the steep down-going section of $MR_b$ (derived from $MR_{b3}$) hits the $Q_b$ axis instead. This gives us the Type II optimum shown in Figure 4, which is drawn on the assumption that $R_1 = 20$, $R_2 = 18$, $k = 25/9$. The reason for the Type I/Type II switch being at $R_2 = (2/3)R_1$ is as follows: $MR_{b2}$ is positive at the left edge of its domain and decreasing in $Q_b$. It therefore reaches its lowest value at the right-hand edge of its domain, that is, where $Q_b = kR_2(R_1 - R_2/2)$. Call this $Q_{b*}$. At $Q_{b*}$, $MR_{b2} = (3/2)R_2 - R_1$. If $MR_{b2}$ is negative at this point, then $MR_{b2}$ must equal
zero somewhere to the left of \( Q^*_b \), which gives a Type I optimum. On the other hand, if \( MR_{b2}(Q^*_b) > 0 \), then \( MR_{b3} \) will intersect the horizontal axis, giving a Type II optimum. So what determines the switch between Type I and Type II optima is the \( R_1 \) and \( R_2 \) pair yielding \( MR_{b2}(Q^*_b) = (3/2)R_2 - R_1 = 0 \), which happens when \( R_2 = (2/3)R_1 \).

There will not be a Type III optimum where \( MR_{b1} = 0 \). This is due to the restrictions on the domain of \( MR_{b1} \) to values of \( Q_b \) below \( \frac{kR_2^2}{2} \). This can be checked by noting that \( MR_{b1} \) is at its lowest value when \( Q_b = \frac{kR_2^2}{2} \), and at this value of \( Q_b \) we have \( MR_{b1}(Q_b) = R_1 + R_2 - \frac{3\sqrt{Q_b}}{\sqrt{2k}} = R_1 + R_2 - \frac{3\sqrt{kR_2^2}}{\sqrt{2k}} = R_1 + R_2 - \frac{3kR_2}{\sqrt{2k}} = R_1 + R_2 - \frac{R_2}{2} \), which is positive because \( R_1 > R_2 \). Hence, \( MR_{b1} \) cannot hit the horizontal axis. The intuition behind this is that as \( P_b \) falls from its height of \( R_1 + R_2 \) at the top left, bundle demand initially rises very fast, and this holds MR above zero.

Some of the properties of the bundling optimum depend upon whether we have a Type I or Type II optimum, that is, they depend upon whether the maximum reservation value of good 2 is high or low relative to the maximum reservation value of good 1. Bundling cheap goods with expensive goods gives different results from bundling goods of roughly equal value. This is a point which has not received any significant attention in the bundling literature. The linear segment of the bundle demand curve lies over the region running from \( Q_b = \frac{kR_2^2}{2} \) on the left to \( Q_b = kR_1 R_2 - \frac{kR_2^2}{2} \) on the right. When \( R_2 \) is small relative to \( R_1 \), the linear segment is long; and when \( R_2 = R_1 \), the linear segment vanishes. We will analyze the two pure bundling optima types in turn.

### 3.2 Type I \((R_2(2/3)R_1)\) Pure Bundling Equilibrium with Zero Cost

In this case, the firm will maximize profit when it sets \( MR_{b2} = 0 \). The quantity of bundle sales will be given by \( Q_{b2} = \frac{1}{2}kR_2(R_1 + \frac{R_2}{2}) \), and the optimum price of the bundle will be \( \frac{R_1}{2} + \frac{R_2}{4} \), which is obviously lower than the sum of the optimal prices when the goods are only sold separately. This yields a simple pricing rule for a firm that switches from independent pricing to bundling—set the price for the bundle equal to the old price of the expensive good (i.e., \( \frac{R_1}{2} \)) plus half of the old price of the inexpensive good (i.e., \( \frac{R_2}{4} \)). The advertisements for the bundle might read, “But wait. Now you can get a single package with both goods at the old price for good 1 plus just half the old price of good 2.”

Consider Figure 5. The nonbundled prices, \( \frac{R_1}{2} \) and \( \frac{R_2}{2} \) are shown for comparison. Note that bundling involves some pluses and minuses.
FIGURE 5. FINDING GAINS AND LOSSES TO CONSUMERS’ SURPLUS AND PROFIT

with respect to consumers’ surplus and the firm’s revenues and profits. Traders in the upper right rectangle, marked 1, would have bought both goods separately at \( R_1 \) and \( R_2 \), but now get them both at a discount, paying only \( \frac{R_1}{2} + \frac{R_2}{4} \). (Note that one-fourth of all \( m \) traders reside in region 1.) This is good for consumers’ surplus and bad for profits. Pure bundling also results in lost sales to traders in regions 3 and 6, who would have bought the goods separately, but now decline to buy the bundle because they do not value the “other” good highly enough to pay the extra cost of the bundle over the old price of the one good they used to buy. This is bad for both consumers’ surplus and profits. On the other hand, bundling results in sales to traders in region 4, who otherwise would not buy anything, and it extracts extra revenue from traders in regions 2a, 2b, 5a, and 5b, who now buy the bundle instead of purchasing only one good. One of the values of having closed-form solutions is that we can now compute the net effects from these conflicting forces on profit and consumers’ surplus.

Because we are at the moment assuming that there are no costs, the net effect of bundling on profits can be found by comparing total revenues in the bundling versus independent sales cases. We turn to this question first.
Total revenue (or profit when cost is zero) with pure bundling will be \( \frac{1}{16} k R_2 (2R_1 + R_2)^2 \). The net change in profits under pure bundling versus selling the products independently will be found by subtracting total revenues with independent sales from the bundling total revenue. This yields

\[
\frac{1}{16} k R_2 (2R_1 + R_2)^2 - \frac{1}{4} k R_1^2 R_2 - \frac{1}{4} k R_1 R_2^2 = \frac{k R_2^3}{16} > 0,
\]

which shows that bundling will lead to an increase in profit when we have a Type I optimum, that is, when \( R_2 < (2/3) R_1 \). Looking at this from a slightly different angle, dividing pure bundling TR by pure separate sales TR, we find:

**Claim 1:** Type I pure bundling offers a revenue advantage over separate sales which ranges from an infinitesimally small percentage when \( R_2 \) is just above zero to 6.67% as \( R_2 \) reaches \( (2/3) R_1 \).

*Proof.* Dividing TR for pure bundling by TR for separate sales yields \((2 + R_2 / R_1)^2 / (4 + R_2 / R_1))\), which is 1 when \( R_2 = 0 \) and rises monotonically to 1.067 as \( R_2 \) approaches \( 2/3 R_1 \). □

Notice that the advantage of pure bundling shrinks to zero as \( R_2 / R_1 \) diminishes toward zero. This will be significant when we consider positive marginal cost.

Does the increase in profit come at the expense of consumers’ surplus? The answer turns out to be, no. In this case, bundling raises both profits and consumers’ surplus. The area under the bundled demand curve from \( Q_b = 0 \) to \( Q_b = \frac{k R_2^2}{2} \), which is where \( MR_{b1} \) meets \( MR_{b2} \), is given by

\[
\int_0^{\frac{k R_2^2}{2}} \left( R_1 + R_2 - \sqrt{\frac{2Q_b}{k}} \right) dQ_b = \frac{1}{6} k (3R_1 R_2^2 + R_2^3),
\]

and the area under the bundled demand curve between \( Q_b = \frac{k R_1 R_2}{4} \) and \( Q_b = \frac{k R_1 R_3}{2} + \frac{k R_2^2}{4} \), which is where \( MR_{b2} \) hits the \( Q_b \) axis, is given by

\[
\int_{\frac{k R_1 R_2}{4} + \frac{k R_2^2}{4}}^{\frac{k R_1 R_3}{2} + \frac{k R_2^2}{4}} \left( R_1 + \frac{R_2}{2} - \frac{Q_b R_2}{k R_2} \right) dQ_b = \frac{1}{32} k R_2 (2R_1 - R_2)(6R_1 + R_2).
\]

If we add these two terms and subtract the total revenue at the Type I optimum, we will get the consumers’ surplus under bundling:
\[
\frac{1}{6}k(3R_1^2 R_2^3 + R_3^3) + \frac{1}{32}kR_2(2R_1 - R_2)(6R_1 + R_2) - \frac{1}{16}kR_2(2R_1 + R_2)^2
\]
\[
= \frac{1}{96}kR_2(12R_1^2 + 12R_1 R_2 + 7R_2^2)
\]

When the goods are sold separately, the total consumers’ surplus is given by \(\frac{1}{8}kR_1^2 R_2 + \frac{1}{8}kR_1 R_2^2\). Because

\[
\frac{1}{96}kR_2(12R_1^2 + 12R_1 R_2 + 7R_2^2) - \frac{1}{8}kR_1^2 R_2 - \frac{1}{8}kR_1 R_2^2 = \frac{7kR_2^3}{96} > 0,
\]

there is a net gain in consumers’ surplus from bundling. This should not come as a complete surprise, because there are more units sold under bundling than under separate pricing, and some buyers, like those in regions 1 and 4 of Figure 5, clearly benefit from the bundling. Traders in region 1 used to buy both goods separately, but now get them as a bundle at lower cost, and traders in region 4 did not buy anything under separate sales, but now buy the bundle. The effect on traders in regions 2 and 5 is mixed. Consider the trader whose reservation price pair puts him at point \(v\) in region 2a. If there were no bundling, he would buy only good 1 and his consumer surplus would be the distance \(vy\). With bundling, he might view his situation as follows: he now gets good 1 at a price of \(P_b\), with a consumer surplus of \(vz\), but he also gets good 2 “for free” getting a consumer surplus of \(v = zw = zu\). Thus, his combined consumer surplus with bundling is \(vu\), which is less than the surplus \(vy\) with separate sales, so bundling has reduced his consumer surplus. This will be true for all traders in region 2a. Now consider the trader at point \(a\) in region 2b. Without bundling her consumer surplus is \(ab\). With bundling it is \(ac + cw = ac + cd = ad > ab\), so bundling is beneficial to her and all other traders in region 2b. Similar arguments could be made with respect to traders in regions 5a and 5b. Notice that traders in 2a and 5a place high values on one good and low values on the other. Bundling would be more advantageous to buyers if these regions were less dense, perhaps due to bivariate normality of reservation values rather than uniform densities. Traders in regions 3 and 6 used to earn a consumers’ surplus when they bought the goods separately, but under pure bundling they buy nothing. The end result is that bundling benefits traders in regions 1, 2b, 4, and 5b, whereas it harms those in 2a, 3, 5a, and 6. The graph is an inefficient means for comparing the magnitudes of the gains and losses, but the derivation above shows that on balance bundling increases consumers’ surplus when we have zero marginal cost and a Type I optimum.
3.3 Type II ($R_2 \geq (2/3)R_1$) Pure Bundling Equilibrium with Zero Cost

When $R_2 \geq (2/3)R_1$, we will have a Type II optimum. At the pure bundling optimum, MRb3 = 0, which implies that $Q_b = (2/3)kR_1R_2$. Recall that $kR_1R_2$ gives the number of buyers, so at the optimum, the firm will sell to two-thirds of the potential customer set. Another way to see this is that the optimum number of bundles will be 33% more than the optimum number of goods sold one at a time before bundling.

The optimum bundle price will be given by

$$\sqrt{\frac{2}{3}}\sqrt{R_1R_2}.$$

It follows that the optimum bundling total revenue will be

$$\frac{2}{3}kR_1R_2\sqrt{\frac{2}{3}}\sqrt{R_1R_2} = \frac{2}{3}\sqrt{\frac{2}{3}k(R_1R_2)^{3/2}}.$$

Subtracting the total revenue from nonbundled sales of goods 1 and 2, we get the change in revenue due to a switch from separate sales to pure bundling

$$\frac{2}{3}\sqrt{\frac{2}{3}k(R_1R_2)^{3/2}} - \frac{1}{4}kR_1^2R_2 - \frac{1}{4}kR_1R_2^2$$

$$= \frac{1}{36}kR_1R_2(8\sqrt{6\sqrt{R_1R_2}} - 9R_1 - 9R_2).$$

Claim 2: When $R_2 \geq (2/3)R_1$, $\frac{kR_1R_2}{36}(8\sqrt{6\sqrt{R_1R_2}} - 9R_1 - 9R_2) > 0$, so pure bundling results in higher profits than does separate sales, and further the revenue advantage of pure bundling over separate sales ranges from 6.7% when $R_2 = 2/3R_1$ to 8.8% when $R_2 = R_1$.

Proof. The ratio of pure bundling to separate sale revenue is $\frac{8\sqrt{\frac{2}{3}kR_1R_2}}{3(R_1+R_2)}$, which is monotonically increasing over the range $R_1 \geq R_2 \geq 2/3R_1$, rising from about 1.067 when $R_2 = 2/3R_1$ to about 1.088 when $R_2 = R_1$. Notice that this gives the percentage advantage of pure bundling over separate sales under Type II to be 6.67–8.8%. Note also the advantage from bundling is maximized when $R_2 = R_1$. Again, this will be significant when we consider nonzero costs.

As with the Type I optimum, we can show that pure bundling will increase the consumers’ surplus. This gives:
Claim 3: When $R_2 \geq \frac{2}{3}R_1$, consumers’ surplus under pure bundling exceeds consumers’ surplus with separate sales.

Proof. See Appendix to Closed-Form Complete.

Summarizing our results from these two subsections, we have now established:

Theorem 1: When marginal costs are zero and reservation values are independent and uniformly distributed, pure bundling will raise both profits and consumers’ surplus when compared with independent sales of the two goods.

Notice that we do not assume that traders’ reservation values are negatively correlated. Both Stigler and Adams and Yellen, using discrete examples, speculated that negative correlation of reservation values is necessary for bundling to dominate separate sales. Our finding lends support to that of Schmalensee (1984) that the negative correlation is not necessary. On the other hand, Schmalensee’s numerical simulations lead him to speculate that “pure bundling apparently always lowers consumers’ surplus” (p. S221), where we find the opposite—although Schmalensee uses normally distributed reservation prices as opposed to uniform distributions here. Salinger, using primarily graphical methods, attempts to compare profits and consumers’ surplus under bundling versus separate sales. But his “aggregate components” method is limited in that it does not allow a comparison of optimal separate sales versus optimal bundling.

Under first degree price discrimination, the seller’s increase in profit comes at the expense of the buyers’ consumer surplus, while Theorem 1 shows that pure bundling (with independent and uniformly distributed reservation prices) increases both consumers’ surplus and profit. In part, this is due to the fact that the total number of units sold is higher under bundling than under separate sales. (Compare separate sales quantities with $Q_b$ under either Type I or Type II optima.) Thus, under bundling, more customers are served, each having a bundle reservation value at least as high as the bundle price. And one-fourth of all $m$ traders (those in region 1 of Figure 5 who would have bought both goods separately) get a discount under pure bundling.

Notice further that our results on optimal pure bundle pricing give operational advice to firms working with linear demand curves—all the pricing manager needs to know is the intercept terms for the two goods’ demand functions, that is, $R_1$ and $R_2$. 
4. Mixed Bundling

We now consider the case where the firm offers both the bundle at price $P_b$ and one or both of the goods separately at $P_1$ and $P_2$. Because pure bundling might be viewed as a constrained case of mixed bundling, it is obvious that profits will be weakly higher with mixed bundling. In fact the following is not hard to demonstrate:

Claim 4: As $R_2$ varies between 0 and $R_1$, the total revenue advantage of mixed bundling over separate sales ranges monotonically from 0 (when $R_2$ is 0) to about 10% (when $R_2 = R_1$), and this always dominates the advantage of pure bundling over separate sales.

Proof. See Appendix to Closed-Form Complete.

But there is much else to explore. To begin, consider Figure 6a, which is drawn with $R_1 > P_b > R_2$ and $R_2 > (1/2)R_1$, which will prove to be significant. Traders whose reservation prices put them in region B will buy the bundle. The area of B is $-\frac{1}{2}(P_1 + P_2 - P_b)^2 + (P_2 - P_b + R_1)(P_1 - P_b + R_2)$. Traders in $G_1$ will buy only good 1, and the area of $G_1$ is $(P_b - P_1)(R_1 - P_1)$. And finally, those in $G_2$ will buy only good 2, the area of which is $(P_b - P_2)(R_2 - P_2)$.

Note that mixed bundling offers some pluses and minuses compared with pure bundling. For example, when good 1 is offered separately at $P_1$, those traders in $G_1$ to the right of the $P_b$ line will decline to buy the bundle, and will now buy only good 1 at a lower price. These traders have a low reservation price for good 2, and the extra cost of the bundle over good 1 alone is not worthwhile to them. On the other hand, new sales (compared with pure bundling at $P_b$) are now made to traders in $G_1$ to the left of the $P_b$ line who will now pay $P_1$ for good 1, where under pure bundling they declined to buy the bundle because of their low reservation price for good 2. Region $G_2$ can be analyzed similarly. In addition, $P_b$ itself will be adjusted (as we shall see) when the firm switches from pure bundling to mixed bundling, and this will involve consumers’ surplus losses for the traders in area B who buy the bundle at the new higher bundle price.

In the present case, the firms total sales revenue from mixed bundling will be given by the continuously differentiable function:

$$TR_{mix} = kP_1(P_b - P_1)(R_1 - P_1) + kP_2(P_b - P_2)(R_2 - P_2)$$

$$+ kP_b \left( -\frac{1}{2}(P_1 + P_2 - P_b)^2 + (P_2 - P_b + R_1)(P_1 - P_b + R_2) \right).$$

But there is a catch with the $TR_{mix}$ function. In certain cases when the firm behaves irrationally, $TR_{mix}$ will mis-state total revenue. For
example, suppose the pricing manager makes a mistake and sets $P_b$ so high that $P_b > P_1 + R_2$. Recall that under mixed bundling $kP_1(P_b - P_1)(R_1 - P_1)$ is used for separate sales revenue of good 1 in the TR$_{mix}$ function. But in our present case, $P_b - P_1 > R_2$, and this would extend the G1 rectangle outside $[0, R_1] \times [0, R_2]$, thus overstating separate sales of good 1. Furthermore, in this case no bundles will be sold, but the TR$_{mix}$ function shows a nonzero bundle sales area. Similar problems occur if $P_b > P_1 + P_2$ or $P_b > P_2 + R_1$.

To remedy this, we limit the domain of $(P_1, P_2, P_b)$ points by imposing a few commonsense restrictions. First, firms set individual and bundle prices so that $P_b \leq P_1 + P_2$, otherwise no one would have reason to buy the bundle. Second, $P_b \geq P_1$ and $P_b \geq P_2$, because
otherwise buyers could get both goods for less than the price of one. (In the border case where, say, \( P_1 = P_b \), the firm would be choosing not to offer good 1 separately, because virtually all buyers would pick the bundle at the same price. As we will see shortly, this is an optimal pricing strategy in certain cases.) Finally, we need to be sure that \( 0 < P_i < R_i, i = 1, 2 \). To incorporate these seller-rationality restrictions, we define the set \( \Phi = \{(P_1, P_2, P_b) \mid P_b \leq P_1 + P_2, P_b \geq P_1, P_b \geq P_2, 0 \leq P_i \leq R_i, i = 1, 2 \} \). (Note that the first and third assumptions earlier require that \( P_b - P_1 \leq R_2 \) and \( P_b - P_2 \leq R_1 \)) This restriction could be imposed by writing \( \min[P_b - P_i, R_j] \) in place of \( P_b - P_i \). And we could impose our other constraints in a similar fashion. But if we limit the domain of our total revenue function to \( \Phi \), this problem will not occur. And the restriction to \( \Phi \) is generally easier to deal with than a function involving \( \min() \) and \( \max() \) conditions. Our \( TR_{\text{max}} \) function will now work fine over the domain \( \Phi \), and we can be sure that no point outside \( \Phi \) will have a higher total revenue than the best point in \( \Phi \).

We can then proceed in three steps: First we take derivatives of \( TR_{\text{max}} \) without regard to \( \Phi \), set them equal to zero, and solve simultaneously for \((P_1, P_2, P_b)\). Second, we eliminate any solutions outside \( \Phi \). Third, we check second-order conditions.

Again we are able to derive closed-form solutions for optimal values for \( P_1, P_2, P_b, Q_1, Q_2, Q_b, \) profit, and consumers’ surplus. This is a novel result in the bundling literature. Indeed Venkatesh and Kamakura (2003) have said, “Mixed bundling does not lend itself to closed-form solution,” although their model used uniform trader density, it was more demanding than the present construct.

To find the optimum values for \( P_1, P_2, \) and \( P_b \), we take the partial derivatives of \( TR_{\text{mix}} \), set them equal to zero and solve the resulting set of simultaneous equations.

\[
\frac{\partial TR_{\text{mix}}}{\partial P_1} = k(P_1 - P_b)(3P_1 - 2R_1) = 0
\]

\[
\frac{\partial TR_{\text{mix}}}{\partial P_2} = k(P_2 - P_b)(3P_2 - 2R_2) = 0
\]

\[
\frac{\partial TR_{\text{mix}}}{\partial P_b} = \frac{1}{2} k(-3P_1^2 - 3P_2^2 + 3P_b^2 + 4P_1R_1 - 4P_bR_1
+ 4P_2R_2 - 4P_bR_2 + 2R_1R_2) = 0.
\]

An immediate implication of the above first-order conditions is that when a firm switches from separate sales to mixed bundling, profits will increase as it increases \( P_i \) and decreases \( P_b \). (Check this by evaluating the first-order conditions at the separate sales values \( P_i = \)
You find \( \partial TR_{mix}/\partial P_i > 0 \) and \( \partial TR_{mix}/\partial P_b < 0 \). But what are the magnitudes of the optimal prices, and what happens with consumers’ surplus? In Appendix, we are able to show:

**Theorem 2:** When marginal costs are zero and reservation values are independent and uniformly distributed, optimum mixed bundling prices are:

\[
P_1 = \frac{2R_1}{3}, \quad P_2 = \frac{2R_2}{3}, \quad P_b = \frac{1}{3}(2R_1 + 2R_2 - \sqrt{2R_1 R_2}) \quad \text{if } R_2 > \frac{1}{2}R_1,
\]

\[
P_1 = \frac{1}{6}(3R_1 + 2R_2), \quad P_2 = \frac{2R_2}{3}, \quad P_b = \frac{1}{6}(3R_1 + 2R_2) \quad \text{if } R_2 \leq \frac{1}{2}R_1.
\]

**Proof.** See Appendix to Closed-Form Complete.

Again, note that these are operational rules. All the marketing boss needs to solve for optimal values of \( P_1, P_2, \) and \( P_b \) are the two individual good demand curve intercepts.

Notice also that if \( R_2 > \frac{1}{2}R_1 \), along with the bundle both goods are offered separately at \( P_i = (2/3)R_i \). But if \( R_2 \leq \frac{1}{2}R_1 \), \( P_1 = P_b \) and in effect only the cheaper good is offered separately, that is, buyers are offered good 1 only in a bundle, but good 2 can be purchased separately. This is “partial mixed bundling,” and it is a phenomenon that has been missed in the literature.

The geometry behind the partial mixed bundling outcome is readily apparent. When \( R_2 \) is low relative to \( R_1 \) and the bundle is priced at \( \frac{1}{6}(3R_1 + 2R_2) \), then setting a separate price for good 1 results in a few sales to traders who would otherwise have bought nothing, but these sales would come at the expense of a large number of former bundle buyers switching from the bundle to good 1 alone. Refer to Figure 6b which depicts the present case. Notice that if the firm introduced a separate sale of good 1 at \( P_1 < P_b \), it would add a small triangle of new good 1 buyers below \( P_b \) (as in \( G_1 \) of Figure 6a), but it would give up the relatively large polygon of traders to the right of \( P_b \) who switch from the bundle to the cheaper good 1 alone.

The intuition behind his result is not hard to find. When two goods with disparate valuations, such as a guitar and guitar strings, are bundled, the price of the bundle is primarily determined by the value of the dominant good. Consider a buyer whose reservation value for the secondary good places her in the top 10% of all buyers for that good, she will not buy the bundle unless her reservation value for the dominant good is also relatively high. She may really want new guitar strings, but she is not going to buy them bundled with a new guitar unless she also places a high value on the guitar, so the seller will make much
more profit from his guitar string division, with negligible loss of guitar sales, if strings are offered separately. (Example: Suppose $R_2 = 0.1 R_1$. Then $P_b = (1/6)(3R_1 + 2R_2) = 0.533 R_1$. A trader will buy the bundle if $r_1 + r_2 > P_b$. Suppose the trader’s $r_2 = 0.9 R_2 = 0.09 R_1$. Then the trader will only buy the bundle if $r_1 > 0.4455 R_1$. That is, to buy the bundle, a trader in the 90th percentile for good 2 also needs to be in at least the 44.55th percentile for good 1. Why not offer the dominant good separately? Although there are no bundle buyers with low reservation values for good 1, there are a good many traders buying the bundle at $P_b = (1/6)(3R_1 + 2R_2)$ who have very low reservation values for good 2. If the firm were to offer good 1 separately at $P_1 < P_b$, all of those traders with $r_2 < P_b - P_1$ would quit buying the bundle and instead buy only good 1 at $P_1$. Even though there would be a few extra sales of good 1, the drop in higher margin bundle sales would keep the firm from offering good 1 separately. (Continuing the previous example, any trader with $r_1 > 0.533 R_1$ would buy the bundle, even with $r_2$ near zero.)

Schmalensee, working with Gaussian distributions of reservation values, obtains a result which “merely suggests that the symmetric case is most favorable to mixed bundling; it does not prove it.” (p. S228) Our results offer a clear picture for the symmetric case with a uniform distribution of reservation values: When $R_2$ is near $R_1$, Schmalensee’s “symmetric” case, mixed bundling dominates and both goods are offered separately as well as in the bundle. When $R_2 < (1/2) R_1$, mixed bundling still dominates, but only the cheaper good is offered separately.

It is interesting to compare the price of the bundle under mixed bundling with the price under pure bundling. In Appendix, we show:

**Theorem 3:** When marginal costs are zero and reservation values are independent and uniformly distributed, the optimum bundle price is greater under mixed bundling than under pure bundling.

Intuitively, the following might offer a rationale for this result. If a pure bundler decides to offer the two goods separately along with the bundle, then the buyers who purchase the separate goods are those with high reservation prices for one good and low reservation prices for the other. They are located at the top left and bottom right of $[0, R_1] \times [0, R_2]$. Some of these buyers would have bought the bundle, but are now “removed” from the distribution after buying good 1 or 2 alone. This has the effect of raising the average valuation of the bundle, and this lets the firm increase its bundle price after beginning separate sales.

In a switch from pure to mixed bundling, some traders are made worse off—those who used to buy the bundle but now buy nothing
and those who continue to buy the bundle at the new higher bundle price. This will factor into our results on consumers’ surplus under mixed bundling. (The value of having a closed-form solution is again apparent. Dansby and Conrad [1984, p. 380], e.g., discuss the welfare implications of pure vs mixed bundling and consider the case where the pure bundle price exceeds the mixed bundle price. We see from the above that a knowledgeable seller will not set those prices.)

We turn now to the subject of consumers’ surplus under mixed bundling versus independent sales of the two goods. We are able to show:

**Theorem 4:** When marginal costs are zero and reservation values are independent and uniformly distributed, mixed bundling results in a lower consumers’ surplus than does separate sales of the two goods.

**Proof.** See Appendix to Closed-Form Complete.

The intuition behind the drop in consumers’ surplus is apparent. Consider the case of \( R_2 < \frac{R_1}{2} \). Under mixed bundling the firm divides its customers into two groups—those who have high reservation values for both goods and those who have high reservation values for good 2 but low reservation values for good 1. The first group buys the bundle, and the second buys good 2 alone. The mixed bundling price for separate purchase of good 2, \( \frac{2}{3}R_2 \), is higher than the individual price under purely separate sales \( \frac{1}{2}R_2 \). And the bundle price, \( \frac{1}{2}R_1 + \frac{1}{3}R_2 \) is only slightly lower than the sum of the individual prices under separate sales \( \frac{1}{2}R_1 + \frac{1}{2}R_2 \). So roughly speaking, people who highly value both goods save \( \frac{1}{6}R_2 \) when they buy the bundle, but people who value primarily good 2 pay more under mixed bundling than under separate sales, and a number of traders drop out of the market entirely when the firm makes the switch from separate sales to mixed bundling.

Combining the results of Theorem 4 with those of Theorem 1, we see that consumers’ surplus is highest under pure bundling, lowest under mixed bundling, and intermediate under separate sales.

**5. Bundling With Positive Marginal Cost**

Assume that the marginal cost of good \( i \) is \( C_i \) and the marginal cost for the bundle is \( C = C_1 + C_2 \). We assume that \( C_i < R_i \), to ensure economically meaningful solutions. If the goods are sold separately, routine calculations give the optimum profit for good \( i \) as \( \frac{1}{4}k(C_i - R_i)^2R_j \), so total profit with separate sales would be \( \frac{1}{4}k((C_1 - R_1)^2R_2 + (C_2 - R_2)^2R_1) \). We have already seen (Theorem 1) that if \( C = 0 \), profit from pure bundling dominates profit from separate sales. We now look into the effect of
positive marginal cost. To limit the length of this section, we will here consider only the Type I case where \( R_2 \leq (2/3)R_1 \). Recall that in this case \( MR_{b2} \) crosses through the horizontal axis as in Figure 3. With pure bundling, it would seem that the profit maximum could occur where either (1) \( C \) intersects the linear section of \( MR \), called \( MR_{b2} \) above, or (2) if \( C \) is relatively high, where \( C \) hits \( MR_{b1} \). But in Appendix, we show:

**Claim 4:** Firms will elect not to bundle when \( C \) is high enough to hit \( MR_{b1} \), so we need only consider the case where \( MR_{b2} \) meets \( C \).

We can now compare the bundling profit with the profit earned from separate sales. \( C \) will intersect \( MR_{b2} \) at the quantity \( \frac{1}{2}k(R_1 + \frac{R_2}{R_1} - C)R_2 \), and this will give rise to a bundle price of \( \frac{1}{4}(2C + 2R_1 + R_2) \). Subtracting variable cost from the total revenue, we get a profit under pure bundling of \( \frac{1}{16}kR_2(2R_1 + R_2 - 2C)^2 \). Profit from optimal bundled sales minus profit from optimal separate sales, what we will call the profit differential, is

\[
\text{Diff} = \frac{1}{16}kR_2(2R_1 + R_2 - 2C_1 - 2C_2)^2 - \frac{1}{4}k((R_1 - C_1)^2R_2 + (R_2 - C_2)^2R_1),
\]

where we make the substitution \( C_1 + C_2 = C \). When marginal cost is high enough, the differential can be negative, that is, separate sales can yield higher profits than bundled sales if the \( C \)'s are sufficiently high. This is obvious from inspection of the above Diff equation, because separate sales can remain positive as long as \( C_i < R_i \) for one of the goods, while that is clearly not the case with bundled sales. In Appendix, we discuss and prove the following result.

**Theorem 5:** If marginal cost is given by \( C_i = hR_i \), there is a well-defined critical bundle marginal cost level, \( C^* = \frac{1}{2}(1 + \frac{R_2}{R_1} - \sqrt{1 + \frac{R_2}{R_1}})R_1 \), at which the profit differential between pure bundling and separate sales is zero. If marginal cost exceeds the critical value, profits are higher with separate sales than with pure bundling.

This phenomenon has been widely discussed in the bundling literature, although results with this specificity have not been found. On a related issue, Schmalensee speculates that “if symmetry [roughly meaning \( R_1 = R_2 \) in this context, JE] does not hold, pure bundling is less likely to be profit or welfare enhancing.” (p. S218) Our results lend support to that conjecture. When \( R_2 / R_1 \) is small, pure bundling ceases to dominate separate sales at very low marginal cost values, and as \( R_2 / R_1 \)
rises, bundling will dominate at marginal costs as high as 0.293R_1. Why? We have already seen that the profit advantage of pure bundling depends upon the ratio of R_2 to R_1. And we just saw that increasing marginal cost works to diminish the advantage of pure bundling. So if the benefit from pure bundling is already small due to the low value of R_2/R_1, even slight values for cost can eliminate any benefits from pure bundling.

We turn now to a much more complex case—mixed bundling with positive marginal cost. In Appendix, we analyze a case where the cost of the bundle is C = C_1 + C, with C_1 = C_2 and R_1 = R_2 = R. The firm then must set bundle and separate sales prices, P_b and P, respectively to maximize:

\[
\pi_{\text{mix}}(P, P_b) = (2P - C)(P_b - P)(R - P) + (P_b - C) \left( -\frac{1}{2}(2P - P_b)^2 + (P - P_b + R)^2 \right)
\]

on condition that \((P, P_b) \in \Gamma = \{(P, P_b) \mid 2P \geq P_b \geq P \geq 0, P < R, \text{ and } P > C/2\}\. The restriction to \(\Gamma\) rules out optima with (i) \(P_b > 2P\), in which case all sales would be separate, (ii) \(P_b < P\), where all sales would be bundles, (iii) \(P \geq R\), where no one would buy separate goods, and (iv) \(P \leq C/2\), which would mean losses on separate sales.

As a visual aid, a sketch of the graph of \(\pi_{\text{mix}}(P, P_b)\) without the restriction to \(\Gamma\) is shown in Figure 7. The “grooves” in the figure lie above the lines where \(P_b = 2P\) and \(P_b = P\) on the boundaries of \(\Gamma\). We can be sure that the function attains its maximum between these grooves. Hence, our locus of interest is the central “lobe.” Suppose for a moment that \(P_b = 2P\). If so, traders are indifferent to buying the bundle and buying the two goods separately. If the firm were to raise \(P_b\) even slightly, all sales would be separate and further increases in \(P_b\) would have no effect upon profits. So if you look along the profit surface as \(P_b\) increases with a given \(P\)-value, you see that after a certain point, profits remain steady. That produces the “tunnel-shaped” segment at the top of Figure 7, leading away from the viewer. A similar argument explains the tunnel at the near right portion of the figure. Our main result is:

**Theorem 6:** For all \(C \in [0, 2R)\), the optimum \((P, P_b)\) lies in \(\Gamma\). That is, mixed bundling profit exceeds profit from either pure bundling or separate sales for all relevant values of \(C\).

**Proof.** See Appendix to Closed-Form Complete.
6. Perfect Correlation of Reservation Values

The case of perfect positive correlation of reservation prices proves to be quite simple. Suppose traders’ reservation values lie on a line running from the origin to the point \((R_1, R_2)\). When the bundle price is set at \(P_b \in [0, R_1 + R_2]\), it is easy to show that the firm will sell \(m(R_1 + R_2 - P_b)\) bundles. Maximizing the value of bundle sales, we get the optimum bundle price of \(P_b = \frac{R_1 + R_2}{2}\). This is exactly equivalent to selling the goods separately at the optimum prices of \(P_1 = (1/2)R_1\) and \(P_2 = (1/2)R_2\), so bundling offers no changes whatever in this case. In fact, the entire outcome is the same in this case whether the firm bundles or does not—the same traders who would buy the bundle would also buy the goods separately if individual sales were their only option.

Assume now that traders’ reservation price points are uniformly distributed along the straight line running from \((0, R_2)\) to \((R_1, 0)\) in Figure 8. This is perfect negative correlation. We first consider pure bundling with zero cost. With \(R_2 \leq P_b < R_1\), total bundle sales, \(Q_b\), and total revenue from bundle sales, \(TR_b\), will be
$Q_b = \frac{m(R_1 - P_b)}{R_1 - R_2}$ and $TR_b = \frac{mP_b(R_1 - P_b)}{R_1 - R_2}$.

(It proves to be much easier and intuitive to work this case using $m$, the total number of potential buyers, rather than $k$, the density of traders.) A first derivative check easily shows that the interior optimum will occur at $P_b = (1/2)R_1$. This will be the optimum pure bundle price as long as $R_2 < (1/2)R_1$. If $R_2 \geq (1/2)R_1$, the optimum pure bundle price is a corner solution at $P_b = R_2$.

Notice that if $R_2 < (1/2)R_1$, the optimum bundle price is the same as the optimum price for good 1 when the goods are sold separately without bundling. So in this case, the optimum strategy amounts to simply giving away good 2 with any “regular purchase” of good 1. How could it be that such a strategy improves profits? The answer lies in the volume of sales. Without bundling, sales of good 1 are $m/2$. With bundling, sales are $\frac{mR_1}{2(R_2 - R_1)} > \frac{m}{2}$. Because the price is the same in either case and the variable costs for both goods are here assumed to be zero, the profit is higher with bundling. This explains a commonly seen marketing tactic: “Buy this, get that free.”

If $R_2 \geq (1/2)R_1$, then the optimum bundle price is $P_b = R_2$. In this case, revenues from bundling will be $mR_2$. Without bundling, revenues will be $\frac{mR_1}{4} + \frac{mR_2}{4}$. As long as $R_2 > (1/3)R_1$, then $mR_2 > \frac{mR_1}{4} + \frac{mR_2}{4}$, so bundling will raise profits in this case also. This yields:

**Theorem 7:** If reservation values are perfectly negatively correlated and production costs are zero, pure bundling raises profits.
We can also analyze the perfect negative correlation case using the bundle demand curve. Consider Figure 9. The bundle demand curve will be linear, running from \((0, R_1)\) to \((m, R_2)\). If \(R_2\) is “low,” as shown on the figure to the left, then \(MR_b\) hits the quantity axis below \(m\), and the optimum bundle price is found at \((1/2)R_1\). If \(R_2\) is “high,” then \(MR_b\) is still positive as it crosses above \(Q_b\) at \(m\). This gives us a corner solution with the optimum \(P_b = R_2\). These figures are handy for analyzing the case of positive marginal cost, but I will leave this as an easy exercise for the reader.

The bundle demand curve approach makes analysis of consumers’ surplus particularly easy. If \(R_2 > (1/2)R_1\) and \(P_b = R_2\), then consumer’s surplus is \(m(R_1 - R_2)\). Comparing this with the combined consumers’ surplus when the goods are sold separately, \(m(R_1 + R_2)/8\), we see that the consumers’ surplus under pure bundling is only greater than the surplus under separate sales if \(R_2 < (3/5)R_1\). Otherwise, bundling reduces the consumers’ surplus. As \(R_2\) approaches \(R_1\), the consumers’ surplus vanishes entirely. If \(R_2 < (1/2)R_1\), then the pure bundling consumers’ surplus will be \(R_1^2m/(8(R_1 - R_2))\), which is greater than the aggregate nonbundled consumers’ surplus. To summarize, we have established:

**Theorem 8:** If reservation values are perfectly negatively correlated and production costs are zero, then pure bundling will increase consumers’ surplus only if \(R_2 < (3/5)R_1\).

Again we find that our results depend upon the relative valuations of the two goods. Much of the literature on bundling has proceeded
on the assumption that $R_1 = R_2$, and this has led to some erroneous conclusions. Both Salinger (1995) and Pindyck and Rubinfeld (2001, p. 395) assert that the firm will extract all of the consumers’ surplus if demands are perfectly negatively correlated. We see from the above that this is only true when $R_1 = R_2$ and that consumers’ surplus is actually increased by pure bundling when $R_2 < (3/5)R_1$.

It should be obvious that mixed bundling can raise profits beyond those attainable with pure bundling in this case. Consider Figure 8 again. The firm could raise profits simply by leaving the bundle price at $P_b$ and adding a separate sale of good 2 at $P_2 = \frac{(R_1 - P_b)R_2}{R_1 - R_2}$. In this way the firm would not cut into existing bundle sales, and it would now sell good 2 to all of the traders on the segment AC. Juggling with $P_b$ and $P_2$ can improve profits further. Notice that we again see the cheaper good offered separately and the more expensive good sold only in the bundle.

We can, if we make a simplifying assumption, get an interesting result for the mixed bundling case with positive marginal costs and perfectly negatively correlated reservation values. Suppose for the moment that $R_1 = R_2 = R$. If marginal costs are zero, the optimum pure bundling price is obviously $R$, because every trader’s reservation price for the bundle is $R$. There is no point in offering either good separately at a price below $P_b = R$, because that would reduce revenues with no favorable effect on cost. Now suppose the marginal costs are $C_1 \geq 0$ and $C_2 \geq 0$. We can show

**Claim 6:** With perfect negative correlation of reservation values, the optimum mixed bundling prices are $P_b = R$, $P_1 = R - (C_2)/2$ and $P_2 = R - (C_1)/2$.

**Proof.** See Appendix to Closed-Form Complete.

It follows that nonzero marginal cost causes a switch from pure bundling to mixed bundling. Note that $P_j$ is a function of $C_i$. It may at first seem odd that $C_i > 0$ causes the firm to offer good $j$ separately, but on reflection this makes excellent sense. If good $i$ is expensive to produce, the firm does not want buyers with low reservation prices for $i$ to get it in the bundle, and the way to properly incentivize these buyers is to offer good $j$ separately at a price below $P_b$. If customers do not value good $i$ sufficiently, they will then buy good $j$ alone.

Salinger comments, “While there is no general proposition that the incentive to bundle is a decreasing function of the correlation between reservation values, the incentive tends to be strongest for negatively correlated demands provided that costs are low.” (p. 95) Our analysis provides some clarification in this area. We have seen that if reservation
values are independently drawn from uniform distributions, then pure bundling dominates separate sales as long as marginal costs are relatively low. If demands are perfectly negatively correlated, we can show that the range of marginal cost values under which pure bundling dominates separate sales is considerably broader. I will describe only the simplest case here. For the moment, let $R_1 = R_2 = R$, assume that the marginal cost of each good is $C/2$ and the marginal cost of the bundle is $C$. Profits under pure bundling will be $(R - C)m$ and optimal total profit with separate sales will be $(m/2R)(R - C/2)^2$. Subtracting separate-sale profit from pure bundling profit, we get $\frac{1}{8}m(4C - \frac{C^2}{R} + 4R)$, which is positive as long as $C/R < 2(-1 + \sqrt{2}) = 0.828427$. So when reservation values are perfectly negatively correlated, $C$ can be as much as about 83% of $R$ with pure bundling dominating separate sales. This is a much stronger result than we achieved in the uniform distribution case, so apparently the incentive to bundle is a decreasing function of the correlation between reservation values, and in the $R_1 = R_2$ case, costs do not have to be “low” for pure bundling to dominate.

7. Abbreviated Results on Complements and Substitutes

Why is it that software is bundled with hardware, or shampoo is bundled with conditioner, but conditioner is not bundled with Rogaine, and software is not bundled with cat food? Intuition suggests that bundling makes more sense when the goods are complements. Until now we have assumed that a trader’s reservation value for the bundle is equal to the sum of her reservation values for goods 1 and 2, that is, $r_1 + r_2$. Our hypothesis has been that the magnitude of $r_i$ does not depend upon whether or not the trader actually buys any of good $j$. It is conventional in the bundling literature to model complements and substitutes by setting the trader’s valuation of the bundle at something other than $r_1 + r_2$. Venkatesh and Kamakura, for example, use $r_b = (1 + \theta)(r_1 + r_2)$, where $\theta > 0$ if the goods are complements, and $\theta < 0$ if the goods are substitutes. But, as I outline in a few moments, if goods $i$ and $j$ are not independent in the mind of the trader, the reservation value for good $i$ should be contingent upon whether or not good $j$ is actually consumed. In a sense, when we move from the assumption of independence of reservation values to the area of complements and substitutes, the whole $r_i$ concept needs to be reevaluated.

The standard way of describing complementary goods is to say that a trader treats good $j$ as a gross complement to good $i$ when an increase in the price of $i$ causes a decrease in his demand for
And goods are perfect complements if a trader’s utility function is of the form $U(Q_1, Q_2) = \min[aQ_1, bQ_2], a, b > 0$. In the bundling literature, we generally consider goods which are bought in quantities of either zero or one, and we typically work with reservation values rather than utility functions, although it is possible to write down a utility function which gives rise to the zero–one purchase outcome and which displays perfect complementarity. Consider $U(Q_1, Q_2) = \min[\max[0, \frac{|Q_1 - 1|}{Q_1}], \max[0, \frac{|Q_2 - 1|}{Q_2}]]$, where $|z|$ is the absolute value of $z$. Suppose that the trader has allocated at most $Y$ dollars of his wealth to the purchase of goods 1 and 2. He wishes to maximize $U()$ subject to $Y \geq P_1Q_1 + P_2Q_2$. Suppose either $Y < P_2$ (so the trader is unable to afford good 2) or may be good 2 is out of stock. For that reason he will choose not to buy any of good 1, that is, his reservation price for good 1 will be zero. On the other hand, if $Y > P_2$, and good 2 is in stock, the trader’s reservation value for good 1 is $Y - P_2$. Thus, with complementarity, the trader’s reservation value for good $i$ is contingent upon the price (and availability, if markets do not clear) of good $j$. Without specifying the contingent circumstance, $r_i$ has no independent meaning. We might write $r_i = r'_i$ if the trader can get good $j$, and $r_i = r''_i$ if the trader cannot get $j$. Then, for example, $r'_i > r''_i$ if the goods are complements. And $r''_i = 0$ if the goods are perfect complements.

Two observations: (i) Because $r_i$ can now move with changes in $P_j$, it is not very convenient to work in the space of reservation price points, as we have done until now. In fact, as we have just seen, the very meaning of the reservation price concept for separate goods is problematic in the context of complements and substitutes. Consider another example: A trader with the perfect complement utility function written earlier has decided to allocate at most $100 of his budget to the consumption of goods 1 and 2. He goes to the market to find only separate sales are possible, $P_1 = 60$, and $P_2 = 30$. He will buy the two goods and be happy to keep $10 change. (We could include “the change” in this utility function, but nothing is gained from writing it down.) But what is his reservation price for good 1? Given that $P_2$ is $30, the most he would be willing to pay for good 1 would be $70. Similarly, given $P_1 = 60, the most he would have been willing to pay for good 2 would be $40. But he is not willing to pay $70 + $40 for the bundle. But then what is the meaning of the reservation price concept if it is not that the trader would spend this amount to buy the good in question? Something is clearly wrong with the reservation price concept for individual goods in the context of complementarity. (ii) To get around this problem, we might define $r_i$ to be $r''_i$, that is, the highest price one would be willing to pay for good $i$ when good $j$ is not available, and define $r_b$ to be the highest price one would be willing to pay for the bundle containing
goods 1 and 2. If the goods are perfect complements, then $r_1'' = r_2'' = 0$ and $r_b > r_1'' + r_2''$. (A key without a lock, and a lock without a key are both worth nothing, but a key with a lock is worth something.) When the goods are less than perfectly complementary, like cookies and milk, each has some value without the other, but together they are worth more than the sum of the isolated parts.

It is easy to ignore these complications, return to our analysis, and look at how things change when $r_b = r_1 + r_2 + H$, where $H > 0$ means the goods are complements and $H < 0$ means substitutes. If we then looked at the case of complements in pure bundling, we would see that the quantity of the bundle demanded at $P_b$ is then the same as the quantity demanded at $P_b - H$ when the goods are independent. Hence, complementarity would simply shift the bundle demand curve (like that found in Figure 3) upward by the amount $H$. This upward shift would increase output and profits, and it would make bundling pay at marginal cost values that would otherwise prompt separate sales. But thoughts on the contingent nature or the $r_i$ give me concerns about proceeding in this facile manner. The topic certainly calls for more work.

8. Conclusion

We set out to find closed-form solutions and “hard answers” to a number of issues in the bundling literature. On this point we can claim partial success. Issues related to prices, profits, and consumers’ surplus have been advanced with respect to both pure and mixed bundling when reservation values are independently drawn from uniform distributions. The cases of perfect positive and negative correlations of reservation values have been worked out. The case of pure bundling with positive marginal cost has also been clarified. And although we have found a simple way to integrate the study of complements and substitutes, that area too could benefit from further study.

Supporting Information

Additional Supporting Information may be found in the online version of this article:

Appendix S1. Contains all of the longer proofs.
Figure S1. Supports the proof of Theorem 4 on mixed bundling.

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References


