Sturm-Liouville Theorems

The Sturm-Liouville Equation: $\left(r(x)X'(x)\right)' + q(x)X(x) + \lambda p(x)X(x) = 0, \quad a \le x \le b$ together with boundary conditions at both a and b. (unless $r(a) = 0$ and/or $r(b) = 0$).

1. All the eigenvalues are real.
2. There exists an infinite sequence, $\lambda_1 < \lambda_2 < \lambda_3 < \ldots \lambda_n < \lambda_{n+1} < \ldots$
   1. There is a smallest eigenvalue: λ_1
   2. There is not a largest eigenvalue: $\lambda_n \to \infty$ as $n \to \infty$
3. Corresponding to each eigenvalue λ_n, there is a corresponding eigenfunction, $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant) and each $\phi_n(x)$ will have exactly n-1 "zeros" for a < x < b.
4. The eigenfunctions form a "complete" set, meaning that any piecewise smooth function, $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions; $f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function, $p(x)$ found in the Sturm-Liouville equation: This means $\int_a^b \phi_n(x)\phi_m(x)p(x)dx = 0$ if $n \ne m$.
6. Any eigenvalue can be related to its eigenfunctions by the so-called Rayleigh Quotient: $$\lambda = \frac{-r(x)X(x)X'(x)\big|_a^b + \int_a^b r(x)\left(X'(x)\right)^2 - q(x)\left(X(x)\right)^2 dx}{\int_a^b \left(X(x)\right)^2 p(x)dx}$$ Where the boundary conditions usually simplify this expression a lot.