Math 135 Final Exam Fall 2001

1. Given \( A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 1 & -2 & 1 & 1 \\ -1 & 2 & 2 & -7 \end{pmatrix} \) and its reduced echelon form, \( R = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \).

   The 4 fundamental subspaces are \( \text{col}(A) \), \( \text{row}(A) \), \( \text{null}(A) \) and \( \text{null}(A^T) \).

   a. Give the dimension of each subspace.

   b. Give a basis for each subspace.

   c. What must "c" be so that \( A \bar{x} = \begin{pmatrix} 2 \\ 5 \\ c \end{pmatrix} \) is solvable?

   d. Find the complete solution to part c. when it is solvable.
2. Let $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$  

SHOW ALL WORK!

a. Why is $A$ guaranteed to have 4 independent eigenvectors?

b. Find the eigenvalues and 4 independent eigenvectors

c. What is the rank of $A + I$?

d. Is $A$ positive definite? Explain.
3. Let a straight line be given by \( y = C + D \cdot t \) and assume that \( y = 0 \) when \( t = 1 \), \( y = 1 \) when \( t = 0 \) and \( y = B \) when \( t = -1 \).

   a. What is the coefficient matrix, \( A \), of the corresponding system:

   \[
   A \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ B \end{pmatrix}
   \]

   b. What conditions will ensure that the system is solvable?

   c. Apply Gram-Schmidt to find \( Q \) in the \( A = QR \) factorization.

   d. Find the projection of \( \begin{pmatrix} 0 \\ 1 \\ B \end{pmatrix} \) onto the column space of \( A \).
4. Gram-Schmidt is \( A = QR \) where we assume the columns of \( A \) are independent (otherwise we would simply throw the redundant columns away).

a. Explain why \( A^T A \) is positive definite (hence the pivots and eigenvalues will be positive).

b. If \( S \) is the subspace spanned by the columns of \( A \), give a formula for the projection matrix, \( P \), that projects onto \( S \). Explain where this formula comes from.

c. If \( U \Sigma V^T = A \) is the singular value decomposition of \( A \), give a formula for the best least squares solution, \( \hat{x} \), to \( A \hat{x} = \hat{b} \) (simplify your answer as much as possible)
a. Explain why every eigenvector of $A$ is either in the column space of $A$ or in the null space of $A$. (or explain why this is false)

b. From $A = S \Lambda S^{-1}$ find the eigenvalue matrix and the eigenvector matrix for $A^T$. How are the eigenvalues of $A$ and $A^T$ related?

c. Suppose $A \vec{x} = 0$ and $A^T \vec{y} = 2 \vec{y}$. Prove that $\vec{x}$ is orthogonal to $\vec{y}$ using subspace ideas from the fundamental theorem of linear algebra and part a.