Limit Concepts and Problems

1. Evaluate the following limits exactly:

\[
\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 3x + 2}, \quad \lim_{x \to 0} \frac{1 - \sqrt{1 - x^2}}{x}, \quad \lim_{x \to 0} \frac{e^x - 1}{\sin x}, \\
\lim_{t \to 2} \frac{\sqrt{t} + 2 - 2}{t - 2}, \quad \lim_{x \to \infty} \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}
\]

2. Answer the following questions concerning the function \( f(x) \) graphed below. If a limit does not exist and the answer is not \( \infty \) or \( -\infty \), write DNE. Note: The dotted lines represent asymptotes.

- (a) \( \lim_{x \to 1^-} f(x) = \)
- (b) \( \lim_{x \to -2^-} f(x) = \)
- (c) \( \lim_{x \to 0^+} f(x) = \)
- (d) \( \lim_{x \to 0^-} f(x) = \)
- (e) \( \lim_{x \to -\infty} f(x) = \)
- (f) Is \( f \) continuous at \(-2\)?

3. At what value of \( x \) does the function

\[ f(x) = \frac{x + 1}{x^2 + x} \]

have

- (a) a removable discontinuity?
- (b) an infinite discontinuity?

Derivative Concepts and Problems

1. Use the definition of derivative to compute:

\[ (\frac{5}{x})', \quad \frac{d}{dx}(2 - x^2), \quad (7 + x - 3x^2)' \]
2. Find the derivatives of the following functions (box your answer).

\[
\begin{align*}
    y &= \frac{2x^6}{3} + \frac{7}{x^2} - 12, \\
    y &= \cos(3x) + \ln(x^3 + 2), \\
    \ln x + \tan(x^2 - 2), \\
    f(x) &= \frac{2e^x}{\tan x - x^2}, \\
    f(x) &= \frac{e^x - 1}{x^3 + \sqrt{x}}, \\
    f(x) &= \sqrt{x}\sin^{-1}(x), \\
    f(t) &= \frac{1}{t}\tan^{-1}(x)
\end{align*}
\]

3. Let \( f \) and \( g \) be differentiable functions and let the values of these functions and their derivatives at the given \( x \) values be given by the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( f'(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>6</td>
<td>11</td>
<td>10</td>
</tr>
</tbody>
</table>

(a) Find \( h'(2) \) where \( h(x) = f(g(x)) \).
(b) Find the derivative of \( fg \) at \( x = 5 \).
(c) Find the derivative of \( f/g \) at \( x = 5 \).

4. Using the table of values for \( u \) and \( v \), given below, find \( w'(0) \) where \( w(x) = u(v(x)) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( u(x) )</th>
<th>( v(x) )</th>
<th>( u'(x) )</th>
<th>( v'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

Related rates/implicit differentiation

1. Find the slope of the graph of the equation, \( x^3 + 2xy^2 = 12 \), at the point \((2, -1)\).
2. Determine the slope of the graph of the equation, \( x^2y^3 + 3y^2 = x + 9y \) at the point \((3,1)\).
3. Water is leaking out of an inverted conical tank at a rate of 0.01m³/min. The tank has height 6 m and the diameter at the top is 4 m. How fast is Is the water level falling when the height of the water is 2 m?
4. If you are given that \( \frac{dS}{dv} = 13.7 \) and the formula, \( S(v) = v^2 - \frac{1}{v} \), find \( \frac{dv}{dw} \) when \( v = 3 \).
5. A 13 ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving away at the rate of 5ft/s. How fast is the top of the ladder sliding down the wall then?
6. The length \( l \) of a rectangle is decreasing at the rate of 2cm/s while the width \( w \) is increasing at the rate of 2cm/s. When \( l = 12 \) cm and \( w = 5 \) cm, find the rates of change of a) the area, b) the perimeter, and c) the lengths of the diagonals of the rectangle.
7. A kite 100 ft above the ground moves horizontally at a speed of 8ft/s. At what rate is the angle between the string and the horizontal distance decreasing when 200 ft of string have been let out?

Basic Max-Min

1. Find the exact \( x \)-coordinates of the absolute maximum and absolute minimum values of \( f(x) = x^2 + 2/x \) on the interval \([\frac{1}{2}, 2]\).

2. Find the exact maximum and minimum values of \( f(x) = \sqrt{9-x^2} - 3 \) on the interval \([-1, 2]\).

3. Find the max and min values of \( f(x) \) and where they occur if \( f(x) = 2 - \frac{1}{x^4} \) on the interval \([\frac{1}{2}, 3]\).

4. Find the dimensions of an open-top rectangular box with volume \( V_0 \) if it is to be made with the least possible material. You may assume that the box has a square base.

5. Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

6. You are designing a rectangular poster to contain 50in.\(^2\) of printing with a 4-in. margin at the top and bottom and a 2-in. margin at the each side. What overall dimensions will minimize the amount of paper used?
Curve Sketching 1. Given the graph of $y = f(x)$ below, circle the letter of the graph that best represents $y = f'(x)$.

[Graphs A, B, C, D are depicted with respective labels A) to D).]
2. Given the graph of $f'$ below, answer the following questions about $f$.

(a) Which is larger $f(x3)$ or $f(x4)$ (Circle one)?
(b) Determine all $x$-coordinates, if any, at which $f$ has a local maximum.
(c) Determine all $x$-coordinates, if any, at which $f$ has a local minimum.
(d) Determine all $x$-coordinates, if any, at which $f$ has an inflection point.

3. Find the asymptotes and relative extrema for

$$y = \frac{x}{(x-1)^2}.$$ 

Newton’s Method/Linear Approximation/Mean Value Theorem

1. Given the following equation $f'(x) = \sqrt{x^3 + 1}$, $f(1) = 2$, use differentials to estimate $f(1.1)$.

2. Find the equation of the tangent line to $f(x) = 1/x$ at $x = 2$.

Answer the following problems using the labeled $x$-values

3. For which starting values will Newton’s Method converge to $x4$? $x7$?

4. Illustrate by drawing on the graph above the result of Newton’s Method one time using $x8$ as the initial input value.

5. For which initial values of $x$ will Newton’s Method fail? Explain or illustrate why the method fails.
6. Illustrate by drawing on the graph above the point that Newton’s method will converge to if the initial starting value is \( x_5 \).

7. Locate on the graph above the point that satisfies the mean value theorem on the interval \([x_3, x_6]\).

8. Find the point “c” that satisfies the mean value theorem for \( f(x) = \sqrt{x + 1} \) on the interval \([3, 8]\).

**Definition and estimation of definite integrals**

1. Express the given limit as a definite integral:

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{2(k \cdot \frac{3}{n})-1} \left( \frac{3}{n} \right) .
\]

2. Express the definite integral as a limit:

\[
\int_{0}^{7} (e^x + 4x^2)dx.
\]

3. State the definition of:

\[
\int_{0}^{4} (\ln(x) - 12)dx.
\]

4. The following graph of \( f \) consists of line segments and semicircles. Use it to evaluate the following integrals:

(a)

\[
\int_{0}^{16} f(x)dx
\]

(b)

\[
\int_{0}^{8} f(x)dx
\]

(c)

\[
\int_{4}^{10} f(x)dx
\]

5. Use the midpoint rule and 4 rectangles to estimate \( \int_{2}^{10} \frac{1}{x^2+1}dx \). You need not simplify your answer.
Multiple Choice Questions

1. Let \( f(x) = 2x - 1 \) on the interval \([0, 3]\). Let the interval be divided into three equal subintervals. Find the value of the Riemann sum \( \sum_{i=1}^{n} f(x_i^*) \Delta x_i \) using the left endpoint of each subinterval.

   \[ \begin{array}{cccc}
   \text{A)} & -1 & \text{B)} & 0 & \text{C)} & 1 & \text{D)} & 3 \\
   \text{E)} & 5 & \text{F)} & 6 & \text{G)} & 7 & \text{H)} & 9 \\
   \end{array} \]

2. Let \( f(x) = (x + 1)^2 \) on the interval \([0, 3]\). Let the interval be divided into three equal subintervals. Find the value of the Riemann sum \( \sum_{i=1}^{n} f(x_i^*) \Delta x_i \) using the midpoint of each subinterval.

   \[ \begin{array}{cccc}
   \text{A)} & 1 & \text{B)} & 4 & \text{C)} & 14 & \text{D)} & 16 \\
   \text{E)} & \frac{63}{4} & \text{F)} & \frac{64}{3} & \text{G)} & 27 & \text{H)} & 29 \\
   \end{array} \]

3. Estimate the area under the graph \( f(x) = 3x^2 + 1 \) from \( x = 0 \) to \( x = 5 \) using five rectangles and right endpoints.

   \[ \begin{array}{cccc}
   \text{A)} & 0 & \text{B)} & 1 & \text{C)} & 5 & \text{D)} & 76 \\
   \text{E)} & 94 & \text{F)} & 125 & \text{G)} & 130 & \text{H)} & 170 \\
   \end{array} \]

4. Let \( f(x) = 4x \) on the interval \([2, 3]\). Let the interval be divided into two equal subintervals. Find the value of the Riemann sum \( \sum_{i=1}^{n} f(x_i^*) \Delta x_i \) if each \( x_i^* \) is the left endpoint of its subinterval.

   \[ \begin{array}{cccc}
   \text{A)} & 8 & \text{B)} & 9 & \text{C)} & \frac{19}{2} & \text{D)} & 10 \\
   \text{E)} & \frac{21}{2} & \text{F)} & 11 & \text{G)} & \frac{29}{2} & \text{H)} & 12 \\
   \end{array} \]

Antiderivatives, the Fundamental Theorem, applications

1. Compute the antiderivative of

\[ f(x) = \sec x \tan x + \sqrt{2}x^2 + \sin x + \frac{2}{\sqrt{1-x^2}} - e^x \]

2. Evaluate the following integrals.

\[ \int \left( \frac{5}{x^2} - \frac{\sqrt{x}}{5} \right) dx, \quad \int_1^4 (4x^3 - 1)dx, \quad \int_3^\pi (x^{-2} - 2x)dx, \quad \int_{\pi/4}^{\pi} \sin x dx, \quad \int_{\pi/4}^{\pi} (3\cos x - 2\sin x)dx \]

3. A particle moves along a line with velocity \( v = 3t - 5 \), \( 0 \leq t \leq 3 \). Find a) the displacement and b) the distance traveled by the particle during the given time period.

4. A particle moves along a line with acceleration \( a(t) = t + 4 \), \( 0 \leq t \leq 10 \) and initial velocity \( v(0) = 5 \). Find a) the velocity at time \( t \) and b) the distance traveled during the given time period.

5. A honey bee population starts with 100 bees and increases at a rate of \( n'(t) \) bees per week. Use calculus to express the total population after 10 years.
6. A particle moves with velocity \( v(t) = t^2 - tm/s \). Circle the answer below that correctly expresses the distance traveled over the first 3 seconds as a definite integral.

A) \( \int_0^3 2t - 1 \, dt \)  
B) \( \int_0^3 t^2 - t \, dt \)  
C) \( \int_0^3 \frac{1}{3} t^3 - \frac{1}{2} t^2 \, dt \)  
D) \( \int_0^3 |2t - 1| \, dt \)  
E) \( \int_0^3 |t^2 - t| \, dt \)  
F) \( \int_0^3 |\frac{1}{3} t^3 - \frac{1}{2} t^2| \, dt \)

7. Let \( F(x) = \int_0^x \sqrt{t^3 + 1} \, dt \). Compute \( F'(x) \).

8. Let \( G(x) = \int_0^{\sin x} t \cos(t^3) \, dt \). Compute \( G'(x) \).

9. Let \( F(x) = \int_0^x f(u) \, du \). Find \( F'(3) \).

10. Let \( G(x) = \int_0^{\sqrt{x}} \frac{s^2}{s^2 + 1} \, ds \). Compute \( G'(x) \).