We also will sample the wide range of applications of such equations. A typical mathematical model of an applied situation will be an initial value problem, consisting of a differential equation of the form in (17) together with an initial condition \( y(x_0) = y_0 \). Note that we call \( y(x_0) = y_0 \) an initial condition whether or not \( x_0 = 0 \). To solve the initial value problem

\[
\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0
\]  

(18)

means to find a differentiable function \( y = y(x) \) that satisfies both conditions in Eq. (18) on some interval containing \( x_0 \).

**Example 10**

Given the solution \( y(x) = 1/(C - x) \) of the differential equation \( dy/dx = y^2 \) discussed in Example 7, solve the initial value problem

\[
\frac{dy}{dx} = y^2, \quad y(1) = 2.
\]

**Solution**

We need only find a value of \( C \) so that the solution \( y(x) = 1/(C - x) \) satisfies the initial condition \( y(1) = 2 \). Substitution of the values \( x = 1 \) and \( y = 2 \) in the given solution yields

\[
y = \frac{1}{C - 1},
\]

so \( 2C - 2 = 1 \), and hence \( C = \frac{3}{2} \). With this value of \( C \) we obtain the desired solution

\[
y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.
\]

Figure 1.1.7 shows the two branches of the graph \( y = 2/(3 - 2x) \). The left-hand branch is the graph on \(( -\infty, \frac{3}{2})\) of the solution of the given initial value problem \( y' = y^2, y(1) = 2 \). The right-hand branch passes through the point \((2, -2)\) and is therefore the graph on \((\frac{3}{2}, \infty)\) of the solution of the different initial value problem \( y' = y^2, y(2) = -2 \).

The central question of greatest immediate interest to us is this: If we are given a differential equation known to have a solution satisfying a given initial condition, how do we actually find or compute that solution? And, once found, what can we do with it? We will see that a relatively few simple techniques—separation of variables (Section 1.4), solution of linear equations (Section 1.5), elementary substitution methods (Section 1.6)—are enough to enable us to solve a variety of first-order equations having impressive applications.

## 1.1 Problems

In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to \( x \).

1. \( y' = 3x^2; y = x^3 + 7 \)
2. \( y' + 2y = 0; y = 3e^{-2x} \)
3. \( y'' + 4y = 0; y_1 = \cos 2x, y_2 = \sin 2x \)
4. \( y'' = 9y; y_1 = e^{3x}, y_2 = e^{-3x} \)
5. \( y' = y + 2e^{-x}; y = e^x - e^{-x} \)
6. \( y'' + 4y'' + 4y = 0; y_1 = e^{-2x}, y_2 = xe^{-2x} \)
7. \( y'' - 2y' + 2y = 0; y_1 = e^x \cos x, y_2 = e^x \sin x \)
8. \( y'' + y = 3 \cos 2x, y_1 = \cos x - \cos 2x, y_2 = \sin x - \cos 2x \)
9. \( y' + 2xy^2 = 0; y = \frac{1}{1 + x^2} \)
10. \( x^2y'' + xy' + y = \ln x; y_1 = x - \ln x, y_2 = \frac{1}{x} - \ln x \)
In Problems 11 through 16, substitute \( y = e^x \) into the given differential equation to determine all values of the constant \( r \) for which \( y = e^x \) is a solution of the equation.

11. \( x^2y'' + 5xy' + 4y = 0 \); \( y_1 = \frac{1}{x^2}, \quad y_2 = \frac{\ln x}{x^2} \)

12. \( xy'' - xy' + 2y = 0 \); \( y_1 = x \cos (\ln x), \quad y_2 = x \sin (\ln x) \)

In Problems 17 through 26, first verify that \( y(x) \) satisfies the given differential equation. Then determine a value of the constant \( C \) so that \( y(x) \) satisfies the given initial condition. Use a computer or graphing calculator (if desired) to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.

17. \( y' + y = 0 \); \( y(x) = Ce^{-x}, \quad y(0) = 2 \)

18. \( y' = 2y; \quad y(x) = Ce^{2x}, \quad y(0) = 3 \)

19. \( y' = y + 1; \quad y(x) = Ce^{x} - 1, \quad y(0) = 5 \)

20. \( y' = y - x; \quad y(x) = Ce^{-x} + x - 1, \quad y(0) = 10 \)

21. \( y' + 3xy = 0; \quad y(x) = Ce^{-x^2}, \quad y(0) = 7 \)

22. \( e^y y' = 1; \quad y(x) = \ln (x + C), \quad y(0) = 0 \)

23. \( x \frac{dy}{dx} + 3y = 2x^2; \quad y(x) = \frac{1}{2}x^2 + Cx^{-3}, \quad y(2) = 1 \)

24. \( x^2y' - 3y = x^3; \quad y(x) = x^2(C + \ln x), \quad y(1) = 17 \)

25. \( y' = 3x^2(y^2 + 1); \quad y(x) = \tan(x^3 + C), \quad y(0) = 1 \)

26. \( y' + y \tan x = \cos x; \quad y(x) = (x + C) \cos x, \quad y(\pi) = 0 \)

In Problems 27 through 31, a function \( y = g(x) \) is described by some geometric property of its graph. Write a differential equation of the form \( \frac{dy}{dx} = f(x, y) \) having the function \( g \) as its solution (or as one of its solutions).

27. The slope of the graph of \( g \) at the point \((x, y)\) is the sum of \( x \) and \( y \).

28. The line tangent to the graph of \( g \) at the point \((x, y)\) intersects the \(x\)-axis at the point \((x/2, 0)\).

29. Every straight line normal to the graph of \( g \) passes through the point \((0, 1)\). Can you guess what the graph of such a function \( g \) might look like?

30. The graph of \( g \) is normal to every curve of the form \( y = x^2 + k \) (a constant) where they meet.

31. The line tangent to the graph of \( g \) at \((x, y)\) passes through the point \((-y, x)\).

In Problems 32 through 36, write—in the manner of Eqs. (3) through (6) of this section—a differential equation that is a mathematical model of the situation described.

32. The time rate of change of a population \( P \) is proportional to the square root of \( P \).

33. The time rate of change of the velocity \( v \) of a coasting motorboat is proportional to the square of \( v \).

34. The acceleration \( dv/dt \) of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.

35. In a city having a fixed population of \( P \) persons, the time rate of change of the number \( N \) of those persons who have heard a certain rumor is proportional to the number of those who have not yet heard the rumor.

36. In a city with a fixed population of \( P \) persons, the time rate of change of the number \( N \) of those persons infected with a certain contagious disease is proportional to the product of the number who have the disease and the number who do not.

In Problems 37 through 42, determine by inspection at least one solution of the given differential equation. That is, use your knowledge of derivatives to make an intelligent guess. Then test your hypothesis.

37. \( y'' = 0 \)

38. \( y' = y \)

39. \( xy' + y = 3x^2 \)

40. \( (y')^2 + y^2 = 1 \)

41. \( y' + y = e^x \)

42. \( y'' + y = 0 \)

43. (a) If \( k \) is a constant, show that a general (one-parameter) solution of the differential equation

\[
\frac{dx}{dt} = kx^3
\]

is given by \( x(t) = 1/(C - kt) \), where \( C \) is an arbitrary constant.

(b) Determine by inspection a solution of the initial value problem \( x' = kx^3, \quad x(0) = 0 \).

44. (a) Continuing Problem 43, assume that \( k \) is positive, and then sketch graphs of solutions of \( x' = kx^3 \) with several typical positive values of \( x(0) \).

(b) How would these solutions differ if the constant \( k \) were negative?

45. Suppose a population \( P \) of rodents satisfies the differential equation \( dP/dt = kP^2 \). Initially, there are \( P(0) = 2 \) rodents, and their number is increasing at the rate of \( dP/dt = 1 \) rodent per month when there are \( P = 10 \) rodents. How long will it take for this population to grow to a hundred rodents? To a thousand? What’s happening here?

46. Suppose the velocity \( v \) of a motorboat coasting in water satisfies the differential equation \( dv/dt = kv^2 \). The initial speed of the motorboat is \( v(0) = 10 \) meters per second (m/s), and \( v \) is decreasing at the rate of \( 1 \) m/s\(^2\) when \( v = 5 \) m/s. How long does it take for the velocity of the boat to decrease to 1 m/s? To \( \frac{1}{10} \) m/s? When does the boat come to a stop?

47. In Example 7 we saw that \( y(x) = 1/(C - x) \) defines a one-parameter family of solutions of the differential equation \( dy/dx = y^2 \). (a) Determine a value of \( C \) so that \( y(10) = 10 \). (b) Is there a value of \( C \) such that \( y(0) = 0 \)? Can you nevertheless find by inspection a solution of \( dy/dx = y^2 \) such that \( y(0) = 0 \)? (c) Figure 1.1.8 shows typical graphs of solutions of the form \( y(x) = 1/(C - x) \). Does it appear that these solution curves fill the entire xy-plane? Can you conclude that, given any point \((a, b)\) in the plane, the differential equation \( dy/dx = y^2 \) has exactly one solution \( y(x) \) satisfying the condition \( y(a) = b \)?
48. (a) Show that \( y(x) = Cx^4 \) defines a one-parameter family of differentiable solutions of the differential equation \( xy' = 4y \) (Fig. 1.1.9). (b) Show that
\[
y(x) = \begin{cases} 
-x^4 & \text{if } x < 0, \\
x^4 & \text{if } x \geq 0
\end{cases}
\]
defines a differentiable solution of \( xy' = 4y \) for all \( x \), but is not of the form \( y(x) = Cx^4 \). (c) Given any two real numbers \( a \) and \( b \), explain why—in contrast to the situation in part (c) of Problem 47—there exist infinitely many differentiable solutions of \( xy' = 4y \) that all satisfy the condition \( y(a) = b \).

1.2 Integrals as General and Particular Solutions

The first-order equation \( dy/dx = f(x, y) \) takes an especially simple form if the right-hand-side function \( f \) does not actually involve the dependent variable \( y \), so
\[
\frac{dy}{dx} = f(x). \tag{1}
\]
In this special case we need only integrate both sides of Eq. (1) to obtain
\[
y(x) = \int f(x) \, dx + C. \tag{2}
\]
This is a general solution of Eq. (1), meaning that it involves an arbitrary constant \( C \), and for every choice of \( C \) it is a solution of the differential equation in (1). If \( G(x) \) is a particular antiderivative of \( f \)—that is, if \( G'(x) \equiv f(x) \)—then
\[
y(x) = G(x) + C. \tag{3}
\]
The graphs of any two such solutions \( y_1(x) = G(x) + C_1 \) and \( y_2(x) = G(x) + C_2 \) on the same interval \( I \) are “parallel” in the sense illustrated by Figs. 1.2.1 and 1.2.2. There we see that the constant \( C \) is geometrically the vertical distance between the two curves \( y(x) = G(x) \) and \( y(x) = G(x) + C \).

To satisfy an initial condition \( y(x_0) = y_0 \), we need only substitute \( x = x_0 \) and \( y = y_0 \) into Eq. (3) to obtain \( y_0 = G(x_0) + C \), so that \( C = y_0 - G(x_0) \). With this choice of \( C \), we obtain the particular solution of Eq. (1) satisfying the initial value problem
\[
\frac{dy}{dx} = f(x), \quad y(x_0) = y_0.
\]
for the swimmer’s trajectory. The initial condition \( y \left(-\frac{1}{2}\right) = 0 \) yields \( C = 1 \), so

\[
y(x) = 3x - 4x^3 + 1.
\]

Then

\[
y \left(\frac{1}{2}\right) = 3 \left(\frac{1}{2}\right) - 4 \left(\frac{1}{2}\right)^3 + 1 = 2,
\]

so the swimmer drifts 2 miles downstream while he swims 1 mile across the river.

### 1.2 Problems

In Problems 1 through 10, find a function \( y = f(x) \) satisfying the given differential equation and the prescribed initial condition.

1. \( \frac{dy}{dx} = 2x + 1; \ y(0) = 3 \)
2. \( \frac{dy}{dx} = (x - 2)^2; \ y(2) = 1 \)
3. \( \frac{dy}{dx} = \sqrt{x}; \ y(4) = 0 \)
4. \( \frac{dy}{dx} = \frac{1}{x^2}; \ y(1) = 5 \)
5. \( \frac{dy}{dx} = \frac{1}{\sqrt{x} + 2}; \ y(2) = -1 \)
6. \( \frac{dy}{dx} = x\sqrt{x^2 + 9}; \ y(-4) = 0 \)
7. \( \frac{dy}{dx} = \frac{10}{x^2 + 1}; \ y(0) = 0 \)
8. \( \frac{dy}{dx} = \cos 2x; \ y(0) = 1 \)
9. \( \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}; \ y(0) = 0 \)
10. \( \frac{dy}{dx} = xe^{-x}; \ y(0) = 1 \)

In Problems 11 through 18, find the position function \( x(t) \) of a moving particle with the given acceleration \( a(t) \), initial position \( x_0 = x(0) \), and initial velocity \( v_0 = v(0) \).

11. \( a(t) = 50, \ v_0 = 10, \ x_0 = 20 \)
12. \( a(t) = -20, \ v_0 = -15, \ x_0 = 5 \)
13. \( a(t) = 3t, \ v_0 = 5, \ x_0 = 0 \)
14. \( a(t) = 2t + 1, \ v_0 = -7, \ x_0 = 4 \)
15. \( a(t) = 4(t + 3)^2, \ v_0 = -1, \ x_0 = 1 \)
16. \( a(t) = \frac{1}{\sqrt{t + 4}}, \ v_0 = -1, \ x_0 = 1 \)
17. \( a(t) = \frac{1}{(t + 1)^2}, \ v_0 = 0, \ x_0 = 0 \)
18. \( a(t) = 50 \sin 5t, \ v_0 = -10, \ x_0 = 8 \)

In Problems 19 through 22, a particle starts at the origin and travels along the x-axis with the velocity function \( v(t) \) whose graph is shown in Figs. 1.2.6 through 1.2.9. Sketch the graph of the resulting position function \( x(t) \) for \( 0 \leq t \leq 10 \).

19. [Figure 1.2.6. Graph of the velocity function \( v(t) \) of Problem 19.]
20. [Figure 1.2.7. Graph of the velocity function \( v(t) \) of Problem 20.]
21. [Figure 1.2.8. Graph of the velocity function \( v(t) \) of Problem 21.]
22. What is the maximum height attained by the arrow of part (b) of Example 3?

23. A ball is dropped from the top of a building 400 ft high. How long does it take to reach the ground? With what speed does the ball strike the ground?

24. The brakes of a car are applied when it is moving at 100 km/h and provide a constant deceleration of 10 meters per second per second (m/s²). How far does the car travel before coming to a stop?

25. A projectile is fired straight upward with an initial velocity of 100 m/s from the top of a building 20 m high and falls to the ground at the base of the building. Find (a) its maximum height above the ground; (b) when it passes the top of the building; (c) its total time in the air.

26. A ball is thrown straight downward from the top of a tall building. The initial speed of the ball is 10 m/s. It strikes the ground with a speed of 60 m/s. How tall is the building?

27. A baseball is thrown straight downward with an initial speed of 40 ft/s from the top of the Washington Monument (555 ft high). How long does it take to reach the ground, and with what speed does the baseball strike the ground?

28. A diesel car gradually speeds up so that for the first 10 s its acceleration is given by

\[ \frac{dv}{dt} = (0.12)t^2 + (0.6)t \text{ (ft/s²)}. \]

If the car starts from rest \((x_0 = 0, v_0 = 0)\), find the distance it has traveled at the end of the first 10 s and its velocity at that time.

29. A car traveling at 60 mi/h (88 ft/s) skids 176 ft after its brakes are suddenly applied. Under the assumption that the braking system provides constant deceleration, what is that deceleration? For how long does the skid continue?

30. The skid marks made by an automobile indicated that its brakes were fully applied for a distance of 75 m before it came to a stop. The car in question is known to have a constant deceleration of 20 m/s² under these conditions. How fast—in km/h—was the car traveling when the brakes were first applied?

31. Suppose that a car skids 15 m if it is moving at 50 km/h when the brakes are applied. Assuming that the car has the same constant deceleration, how far will it skid if it is moving at 100 km/h when the brakes are applied?

32. On the planet Gzyx, a ball dropped from a height of 20 ft hits the ground in 2 s. If a ball is dropped from the top of a 200-ft-tall building on Gzyx, how long will it take to hit the ground? What speed will it hit?

33. A person can throw a ball straight upward from the surface of the earth to a maximum height of 144 ft. How high could this person throw the ball on the planet Gzyx of Problem 29?

34. A stone is dropped from rest at an initial height \(h\) above the surface of the earth. Show that the speed with which it strikes the ground is \(v = \sqrt{2gh}\).

35. Suppose a woman has enough "spring" in her legs to jump (on earth) from the ground to a height of 2.25 ft. If she jumps straight upward with the same initial velocity on the moon—where the surface gravitational acceleration is (approximately) 5.3 ft/s²—how high above the surface will she rise?

36. At noon a car starts from rest at point \(A\) and proceeds at constant acceleration along a straight road toward point \(B\). If the car reaches \(B\) at 12:50 P.M. with a velocity of 60 mi/h, what is the distance from \(A\) to \(B\)?

37. At noon a car starts from rest at point \(A\) and proceeds with constant acceleration along a straight road toward point \(C\), 35 miles away. If the constantly accelerated car arrives at \(C\) with a velocity of 60 mi/h, at what time does it arrive at \(C\)?

38. If \(a = 0.5\) mi and \(v_0 = 9\) mi/h as in Example 4, what must the swimmer's speed \(v_s\) be in order that he drifts only 1 mile downstream as he crosses the river?

39. Suppose that \(a = 0.5\) mi, \(v_0 = 9\) mi/h, and \(v_s = 3\) mi/h as in Example 4, but that the velocity of the river is given by the fourth-degree function

\[ v_r = v_0 \left(1 - \frac{x^2}{a^2}\right) \]

rather than the quadratic function in Eq. (18). Now find how far downstream the swimmer drifts as he crosses the river.

40. A bomb is dropped from a helicopter hovering at an altitude of 800 feet above the ground. From the ground directly beneath the helicopter, a projectile is fired straight upward toward the bomb, exactly 2 seconds after the bomb is released. With what initial velocity should the projectile be fired, in order to hit the bomb at an altitude of exactly 400 feet?

41. A spacecraft is in free fall toward the surface of the moon at a speed of 1000 mph (mi/h). Its retrorockets, when fired, provide a constant deceleration of 20,000 mi/h². At what height above the lunar surface should the astronauts fire the retrorockets to ensure a soft touchdown? (As in Example 2, ignore the moon's gravitational field.)